MATH 2010 Advanced Calculus

Suggested Solution of Homework 9

Section14.8 Q2

$\nabla f = y\mathbf{i} + x\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow y\mathbf{i} + x\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) \Rightarrow y = 2x\lambda$ and $x = 2y\lambda$ $\Rightarrow x = 4x\lambda^2 \Rightarrow x = 0$ or $\lambda = \pm\frac{1}{2}$.

CASE 1: If $x = 0$, then $y = 0$. But $(0, 0)$ is not on the circle $x^2 + y^2 - 10 = 0$ so $x \neq 0$.

CASE 2: $x \neq 0 \Rightarrow \lambda = \pm\frac{1}{2} \Rightarrow y = 2x\left(\pm\frac{1}{2}\right) = \pm x \Rightarrow x^2 + (\pm x)^2 - 10 = 0 \Rightarrow x = \pm\sqrt{5} \Rightarrow y = \pm\sqrt{5}$.

Therefore f takes on its extreme values at $\left(\pm\sqrt{5}, \sqrt{5}\right)$ and $\left(\pm\sqrt{5}, -\sqrt{5}\right)$. The extreme values of f on the circle are 5 and -5.

Section14.8 Q3

$\nabla f = -2x\mathbf{i} - 2y\mathbf{j}$ and $\nabla g = \mathbf{i} + 3\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow -2x\mathbf{i} - 2y\mathbf{j} = \lambda(\mathbf{i} + 3\mathbf{j}) \Rightarrow x = -\frac{\lambda}{2}$ and $y = -\frac{3\lambda}{2}$ $\Rightarrow \left(-\frac{\lambda}{2}\right) + 3\left(-\frac{3\lambda}{2}\right) = 10 \Rightarrow \lambda = -2 \Rightarrow x = 1$ and $y = 3 \Rightarrow f$ takes on its extreme value at $(1, 3)$ on the line. The extreme value is $f(1, 3) = 49 - 1 - 9 = 39$.

Section14.8 Q5

We optimize $f(x, y) = x^2 + y^2$, the square of the distance to the origin, subject to the constraint $g(x, y) = xy^2 - 54 = 0$. Thus $\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$ and $\nabla g = y^2\mathbf{i} + 2xy\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x\mathbf{i} + 2y\mathbf{j}$ $= \lambda\left(y^2\mathbf{i} + 2xy\mathbf{j}\right) \Rightarrow 2x = \lambda y^2$ and $2y = 2\lambda xy$.

CASE 1: If $y = 0$, then $x = 0$. But $(0, 0)$ does not satisfy the constraint $xy^2 = 54$ so $y \neq 0$.

CASE 2: If $y \neq 0$, then $2 = 2\lambda x \Rightarrow x = \frac{1}{\lambda} \Rightarrow 2\left(\frac{1}{\lambda}\right) = \lambda y^2 \Rightarrow y^2 = \frac{2}{\lambda^2}$. Then $xy^2 = 54 \Rightarrow \left(\frac{1}{\lambda}\right)\left(\frac{2}{\lambda^2}\right) = 54$

$$\Rightarrow \lambda^3 = \frac{1}{27} \Rightarrow \lambda = \frac{1}{3} \Rightarrow x = 3 \text{ and } y^2 = 18 \Rightarrow x = 3 \text{ and } y = \pm 3\sqrt{2}.$$

Therefore $\left(3, \pm 3\sqrt{2}\right)$ are the points on the curve $xy^2 = 54$ nearest the origin (since $xy^2 = 54$ has points increasingly far away as y gets close to 0, no points are farthest away).

Section14.8 Q9

$V=\pi r^2 h \Rightarrow 16\pi=\pi r^2 h \Rightarrow 16=r^2 h \Rightarrow g(r,h)=r^2 h-16;\ S=2\pi rh+2\pi r^2 \Rightarrow \nabla S=(2\pi h+4\pi r)\mathbf{i}+2\pi r\mathbf{j}$ and $\nabla g=2rh\mathbf{i}+r^2\mathbf{j}$ so that $\nabla S=\lambda\nabla g \Rightarrow (2\pi rh+4\pi r)\mathbf{i}+2\pi r\mathbf{j}=\lambda\left(2rh\mathbf{i}+r^2\mathbf{j}\right) \Rightarrow 2\pi rh+4\pi r=2rh\lambda$ and $2\pi r=\lambda r^2 \Rightarrow r=0$ or $\lambda=\frac{2\pi}{r}$. But $r=0$ gives no physical can, so $r\neq 0 \Rightarrow \lambda=\frac{2\pi}{r} \Rightarrow 2\pi h+4\pi r=2rh\left(\frac{2\pi}{r}\right)$ $\Rightarrow 2r=h \Rightarrow 16=r^2(2r) \Rightarrow r=2 \Rightarrow h=4;$ thus $r=2$ cm and $h=4$ cm give the only extreme surface area of $24\pi\ \text{cm}^2$. Since $r=4$ cm and $h=1$ cm $\Rightarrow V=16\pi\ \text{cm}^3$ and $S=40\pi\ \text{cm}^2$, which is a larger surface area, then $24\pi\ \text{cm}^2$ must be the minimum surface area.

Section14.8 Q11

$A=(2x)(2y)=4xy$ subject to $g(x,y)=\frac{x^2}{16}+\frac{y^2}{9}-1=0;\ \nabla A=4y\mathbf{i}+4x\mathbf{j}$ and $\nabla g=\frac{x}{8}\mathbf{i}+\frac{2y}{9}\mathbf{j}$ so that $\nabla A=\lambda\nabla g \Rightarrow 4y\mathbf{i}+4x\mathbf{j}=\lambda\left(\frac{x}{8}\mathbf{i}+\frac{2y}{9}\mathbf{j}\right) \Rightarrow 4y=\left(\frac{x}{8}\right)\lambda$ and $4x=\left(\frac{2y}{9}\right)\lambda \Rightarrow \lambda=\frac{32y}{x}$ and $4x=\left(\frac{2y}{9}\right)\left(\frac{32y}{x}\right)$ $\Rightarrow y=\pm\frac{3}{4}x \Rightarrow \frac{x^2}{16}+\frac{\left(\pm\frac{3}{4}x\right)^2}{9}=1 \Rightarrow x^2=8 \Rightarrow x=\pm 2\sqrt{2}$. We use $x=2\sqrt{2}$ since x represents distance. Then $y=\frac{3}{4}\left(2\sqrt{2}\right)=\frac{3\sqrt{2}}{2}$, so the length is $2x=4\sqrt{2}$ and the width is $2y=3\sqrt{2}$.

Section14.8 Q13

$\nabla f=2x\mathbf{i}+2y\mathbf{j}$ and $\nabla g=(2x-2)\mathbf{i}+(2y-4)\mathbf{j}$ so that $\nabla f=\lambda\nabla g=2x\mathbf{i}+2y\mathbf{j}=\lambda\left[(2x-2)\mathbf{i}+(2y-4)\mathbf{j}\right]$ $\Rightarrow 2x=\lambda(2x-2)$ and $2y=\lambda(2y-4) \Rightarrow x=\frac{\lambda}{\lambda-1}$, and $y=\frac{2\lambda}{\lambda-1}, \lambda\neq 1 \Rightarrow y=2x$ $\Rightarrow x^2-2x+(2x)^2-4(2x)=0 \Rightarrow x=0$ and $y=0$, or $x=2$ and $y=4$. $f(0,0)=0$ is the minimum value and $f(2,4)=20$ is the maximum value. (Note that $\lambda=1$ gives $2x=2x-2$ or $0=-2$, which is impossible.)

Section14.8 Q14

$\nabla f=3\mathbf{i}-\mathbf{j}$ and $\nabla g=2x\mathbf{i}+2y\mathbf{j}$ so that $\nabla f=\lambda\nabla g \Rightarrow 3=2\lambda x$ and $-1=2\lambda y \Rightarrow \lambda=\frac{3}{2x}$ and $-1=2\left(\frac{3}{2x}\right)y$ $\Rightarrow y=-\frac{x}{3} \Rightarrow x^2+\left(-\frac{x}{3}\right)^2=4 \Rightarrow 10x^2=36 \Rightarrow x=\pm\frac{6}{\sqrt{10}} \Rightarrow x=\frac{6}{\sqrt{10}}$ and $y=-\frac{2}{\sqrt{10}}$, or $x=-\frac{6}{\sqrt{10}}$ and $y=\frac{2}{\sqrt{10}}$. Therefore $f\left(\frac{6}{\sqrt{10}},-\frac{2}{\sqrt{10}}\right)=\frac{20}{\sqrt{10}}+6=2\sqrt{10}+6\approx 12.325$ is the maximum value, and $f\left(-\frac{6}{\sqrt{10}},\frac{2}{\sqrt{10}}\right)$ $=-2\sqrt{10}+6\approx -0.325$ is the minimum value.

Section14.9 Q2

$$f(x,y)=e^x\cos y \Rightarrow f_x=e^x\cos y,\ f_y=-e^x\sin y,\ f_{xx}=e^x\cos y,\ f_{xy}=-e^x\sin y,\ f_{yy}=-e^x\cos y$$

$$\Rightarrow f(x,y)\approx f(0,0)+xf_x(0,0)+yf_y(0,0)+\tfrac{1}{2}\left[x^2f_{xx}(0,0)+2xyf_{xy}(0,0)+y^2f_{yy}(0,0)\right]$$

$$=1+x\cdot 1+y\cdot 0+\tfrac{1}{2}\left[x^2\cdot 1+2xy\cdot 0+y^2\cdot(-1)\right]=1+x+\tfrac{1}{2}\left(x^2-y^2\right),\ \text{quadratic approximation;}$$

Section14.9 Q5

$$f(x,y)=e^x\ln(1+y)\Rightarrow f_x=e^x\ln(1+y),\ f_y=\tfrac{e^x}{1+y},\ f_{xx}=e^x\ln(1+y),\ f_{xy}=\tfrac{e^x}{1+y},\ f_{yy}=-\tfrac{e^x}{(1+y)^2}$$

$$\Rightarrow f(x,y)\approx f(0,0)+xf_x(0,0)+yf_y(0,0)+\tfrac{1}{2}\left[x^2f_{xx}(0,0)+2xyf_{xy}(0,0)+y^2f_{yy}(0,0)\right]$$

$$=0+x\cdot 0+y\cdot 1+\tfrac{1}{2}\left[x^2\cdot 0+2xy\cdot 1+y^2\cdot(-1)\right]=y+\tfrac{1}{2}\left(2xy-y^2\right),\ \text{quadratic approximation;}$$

Section14.9 Q8

$$f(x,y)=\cos\left(x^2+y^2\right)\Rightarrow f_x=-2x\sin\left(x^2+y^2\right),\ f_y=-2y\sin\left(x^2+y^2\right),$$

$$f_{xx}=-2\sin\left(x^2+y^2\right)-4x^2\cos\left(x^2+y^2\right),\ f_{xy}=-4xy\cos\left(x^2+y^2\right),\ f_{yy}=-2\sin\left(x^2+y^2\right)-4y^2\cos\left(x^2+y^2\right)$$

$$\Rightarrow f(x,y)\approx f(0,0)+xf_x(0,0)+yf_y(0,0)+\tfrac{1}{2}\left[x^2f_{xx}(0,0)+2xyf_{xy}(0,0)+y^2f_{yy}(0,0)\right]$$

$$=1+x\cdot 0+y\cdot 0+\tfrac{1}{2}[x^2\cdot 0+2xy\cdot 0+y^2\cdot 0]=1,\ \text{quadratic approximation;}$$