Thm (Implicit Function Theorem)

Let $\Omega \subseteq \mathbb{R}^{n+k}$ be open, $\vec{F}: \Omega \to \mathbb{R}^k$, $\vec{F} = \begin{bmatrix} F_1 \\ \vdots \\ F_k \end{bmatrix}$ be C^1

Denote $\vec{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$ & $\vec{y} = (y_1, \cdots, y_k) \in \mathbb{R}^k$.

$$\vec{F}(\vec{x}, \vec{y}) = \begin{bmatrix} F_1(\vec{x}, \vec{y}) \\ \vdots \\ F_k(\vec{x}, \vec{y}) \end{bmatrix} = \begin{bmatrix} F_1(x_1, \cdots, x_n, y_1, \cdots, y_k) \\ \vdots \\ F_k(x_1, \cdots, x_n, y_1, \cdots, y_k) \end{bmatrix}$$

Suppose $(\vec{a}, \vec{b}) \in \Omega$, where $\vec{a} \in \mathbb{R}^n$, $\vec{b} \in \mathbb{R}^k$ such that

$$\vec{F}(\vec{a}, \vec{b}) = \vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} \in \mathbb{R}^k$$

and the $k \times k$ matrix

$$\left[\frac{\partial F_i}{\partial y_j}(\vec{a}, \vec{b})\right]_{1 \le i, j \le k} = \begin{bmatrix} \frac{\partial F_1}{\partial y_1}(\vec{a}, \vec{b}) & \cdots & \frac{\partial F_1}{\partial y_k}(\vec{a}, \vec{b}) \\ \vdots & & \vdots \\ \frac{\partial F_k}{\partial y_1}(\vec{a}, \vec{b}) & \cdots & \frac{\partial F_k}{\partial y_k}(\vec{a}, \vec{b}) \end{bmatrix}$$

is invertible (i.e. $\det\left[\frac{\partial F_i}{\partial y_j}(\vec{a}, \vec{b})\right] \neq 0$)

Then there are open sets $U \subseteq \mathbb{R}^n$ containing $\vec{a}$, and $V \subseteq \mathbb{R}^k$ containing $\vec{b}$ such that there exists a unique function $\vec{\varphi}: U \to V$ with $\vec{\varphi}(\vec{a}) = \vec{b}$ and

$$\vec{F}(\vec{x}, \vec{\varphi}(\vec{x})) = \vec{c}, \quad \forall \vec{x} \in U$$

Moreover, $\vec{\varphi}$ is C^1 and (by implicit differentiation)

$$\left[\frac{\partial \varphi_j}{\partial x_\ell}(\vec{x})\right]_{k \times n} = -\left[\frac{\partial F_i}{\partial y_j}(\vec{x}, \vec{\varphi}(\vec{x}))\right]_{k \times k}^{-1} \left[\frac{\partial F_i}{\partial x_\ell}(\vec{x}, \vec{\varphi}(\vec{x}))\right]_{k \times n}$$

(Pf: in MATH3060)

<u>Special case</u> (A): $k=1$ (1 constraint)

$F: \Omega \subset \mathbb{R}^{n+1} \to \mathbb{R}$

$F(\vec{x}, \vec{y}) = F(x_1, \cdots, x_n, y) = c$ (1 constraint)

Suppose $\vec{a} = (a_1, \cdots, a_n) \in \mathbb{R}^n$, $b \in \mathbb{R}$ s.t

$F(a_1, \cdots, a_n, b) = c$

<u>IFT</u> If $\frac{\partial F}{\partial y}(a_1, \cdots, a_n, b) \neq 0$,

then $\exists\ U^{\text{open} \subseteq \mathbb{R}^n}$ s.t. $(a_1, \cdots, a_n) \in U$ and

$V^{\text{open} \subseteq \mathbb{R}}$ s.t. $b \in V$

and $\exists$ unique C^1 $\varphi: U \to V$ s.t. $\varphi(a_1, \cdots, a_n) = b$ &

$F(x_1, \cdots, x_n, \varphi(x_1, \cdots, x_n)) = c$, $\forall (x_1, \cdots, x_n) \in U$

(i.e. $y = \varphi(x_1, \cdots, x_n)$ solves the constraint $F(x_1, \cdots, x_n, y) = c$
"near" $(a_1, \cdots, a_n, b)$)

(Moreover, $\frac{\partial \varphi}{\partial x_i}$ can be calculated using implicit differentiation)

In eg2: $x^2+y^2+z^2=2$ Solve $z=z(x,y)$

(x,y,z) $\mathbb{R}^3$ notation	(x_1,x_2,y) general notation (this "y" is not the "y" on the other side)
$g(x,y,z)=x^2+y^2+z^2=2$ near $(0,1,1)$	$F(x_1,x_2,y)=x_1^2+x_2^2+y^2=c$ $(c=2)$ $\vec{a}=(a_1,a_2)=(0,1)$; $b=1$
$\frac{\partial g}{\partial z}(0,1,1)=2\neq 0$ By IFT $\exists$ $z=z(x,y)$ "near" $(0,1)$ s.t. $\begin{cases} g(x,y,z(x,y))=2 \\ z(0,1)=1 \end{cases}$ $\left(x^2+y^2+(z(x,y))^2=2\right)$	$\frac{\partial F}{\partial y}(a_1,a_2,b)=2\neq 0$ By IFT $\exists$ $y=\varphi(x_1,x_2)$ "near" (a_1,a_2,b) s.t. $\begin{cases} F(x_1,x_2,\varphi(x_1,x_2))=c \\ \varphi(a_1,a_2)=b \end{cases}$ $\uparrow$ $\varphi(0,1)=1$
$\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ can be computed by implicit differentiation	$\frac{\partial \varphi}{\partial x_1}$, $\frac{\partial \varphi}{\partial x_2}$ can be computed by implicit differentiation

Special case (B) $n=1$, $k=2$ (2-constraints)

$$\vec{F} : \Omega \subseteq \mathbb{R}^{1+2} \longrightarrow \mathbb{R}^2$$

$$\vec{F}(x, y_1, y_2) = \begin{bmatrix} F_1(x, y_1, y_2) \\ F_2(x, y_1, y_2) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \vec{c}$$

Suppose (a, b_1, b_2) satisfies the constraints $\vec{F}(a, b_1, b_2) = \vec{c}$

then IFT means

if $\begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} \end{bmatrix}(a, b_1, b_2)$ is invertible (ie. $\det \neq 0$)

then $\exists$ $y_1 = \varphi_1(x)$ & $y_2 = \varphi_2(x)$ "near" (a, b_1, b_2)

solving the constraints (locally)

$$\begin{cases} F_1(x, \varphi_1(x), \varphi_2(x)) = c_1 \\ F_2(x, \varphi_1(x), \varphi_2(x)) = c_2 \end{cases}$$

& $\begin{cases} \varphi_1(a) = b_1 \\ \varphi_2(a) = b_2 \end{cases}$

eg3 $\begin{cases} x^2+y^2+z^2=2 \\ x+z=1 \end{cases}$ Solve for $y=y(x)$, $z=z(x)$? near $(0,1,1)$

(x,y,z) $\longleftrightarrow$ (x, y_1, y_2)

$\mathbb{R}^3$ notation — General Notation

$g(x,y,z) = x^2+y^2+z^2 = 2$

$h(x,y,z) = x+z = 1$

near $(0,1,1)$

$$\begin{bmatrix} \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{bmatrix}_{(0,1,1)} = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$$

invertible ($\det = 2 \neq 0$)

By IFT

$\exists$ $y=y(x)$, $z=z(x)$ "near" $(0,1,1)$ s.t.

$$\begin{cases} g(x, y(x), z(x)) = 2 \\ h(x, y(x), z(x)) = 1 \\ y(0)=1 \\ z(0)=1 \end{cases}$$

$$\left(\begin{cases} x^2+(y(x))^2+(z(x))^2 = 2 \\ x+z(x) = 1 \end{cases} \right)$$

Remark: $\left.\frac{dy}{dx}, \frac{dz}{dx}\right|_{x=0}$ can be calculated by implicit differentiation

$F_1(x,y_1,y_2) = x^2+y_1^2+y_2^2 = c_1 \ (=2)$

$F_2(x,y_1,y_2) = x+y_2 = c_2 \ (=1)$

$a=0$, $\vec{b}=(b_1,b_2)=(1,1)$ $\vec{c}=\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} \end{bmatrix}_{(a,b_1,b_2)} = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$$

invertible ($\det = 2 \neq 0$)

By IFT

$\exists$ $y_1=\varphi_1(x)$, $y_2=\varphi_2(x)$ "near" (a,b_1,b_2) s.t.

$$\begin{cases} F_1(x, \varphi_1(x), \varphi_2(x)) = c_1 \\ F_2(x, \varphi_1(x), \varphi_2(x)) = c_2 \\ \varphi_1(a)=b_1 \\ \varphi_2(a)=b_2 \end{cases}$$

Remark: $\left.\frac{d\varphi_1}{dx}, \frac{d\varphi_2}{dx}\right|_{x=a}$ can be calculated by implicit differentiation eg: Consider the constraints

$$\begin{cases} xz + \sin(yz - x^2) = 8 \\ x + 4y + 3z \quad = 18 \end{cases}$$

$(2,1,4)$ is a solution.

Can we solve 2 of the variables as functions of the remaining variable?

Soln:

$$\vec{F}(x,y,z) = \begin{bmatrix} F_1(x,y,z) \\ F_2(x,y,z) \end{bmatrix} = \begin{bmatrix} xz + \sin(yz - x^2) \\ x + 4y + 3z \end{bmatrix}$$

$$D\vec{F} = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{bmatrix}$$

$$= \begin{bmatrix} z - 2x\cos(yz - x^2) & z\cos(yz - x^2) & x + y\cos(yz - x^2) \\ 1 & 4 & 3 \end{bmatrix}$$

$\Rightarrow$

$$D\vec{F}(2,1,4) = \begin{bmatrix} 0 & 4 & 3 \\ 1 & 4 & 3 \end{bmatrix} \qquad \text{(check!)}$$

$$\begin{array}{cc} x & y \\ \downarrow & \downarrow \end{array} \qquad \begin{array}{cc} x & z \\ \downarrow & \downarrow \end{array} \qquad \begin{array}{cc} y & z \\ \downarrow & \downarrow \end{array}$$

$$\det\begin{pmatrix} 0 & 4 \\ 1 & 4 \end{pmatrix} = -4 \neq 0 \qquad \det\begin{pmatrix} 0 & 3 \\ 1 & 3 \end{pmatrix} = -3 \neq 0 \qquad \det\begin{pmatrix} 4 & 3 \\ 4 & 3 \end{pmatrix} = 0$$

IFT $\Rightarrow$

- x, y can be solved as (diff.) functions of z near $(2,1,4)$

- x, z can be solved as (diff.) functions of y near $(2,1,4)$

- <u>No conclusion</u> on whether y, z can be solved as (diff.) functions of x near $(2,1,4)$

  (In fact, implicit diff. $\Rightarrow$ if $y(x)$, $z(x)$ exists (& diff.)
  then $\begin{bmatrix} 4 & 3 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ which is a contradiction.)

Thm (Inverse Function Theorem)

Let $\vec{f}: \Omega \to \mathbb{R}^n$ be C^1, ($\Omega \subset \mathbb{R}^n$, open)

Suppose $D\vec{f}(\vec{a})$ is invertible ($n \times n$ matrix)

Then $\exists$ open sets $U \subseteq \mathbb{R}^n$ containing $\vec{a}$,

$V \subseteq \mathbb{R}^n$ containing $\vec{b} = \vec{f}(\vec{a})$

such that $\exists$ a unique function

$$\vec{g}: V \to U \quad \text{with}$$

$$\vec{g}(\vec{b}) = \vec{a}$$

satisfying $\begin{cases} \vec{g}(\vec{f}(\vec{x})) = \vec{x}, & \forall \vec{x} \in U \\ \vec{f}(\vec{g}(\vec{y})) = \vec{y}, & \forall \vec{y} \in V \end{cases}$ (i.e. $\vec{g} = (\vec{f}|_U)^{-1}$)

Moreover, $\vec{g}$ is C^1 and

$$D\vec{g}(\vec{y}) = \left[D\vec{f}(\vec{g}(\vec{y}))\right]^{-1}, \quad \forall \vec{y} \in V.$$

(Pf: MATH3060)

Remark: $\vec{g} = (\vec{f}|_U)^{-1}$ is called a local inverse of $\vec{f}$ at $\vec{a}$.

eg: $\vec{f}: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad \vec{f}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 - y^2 \\ 2xy \end{bmatrix}$

Clearly, $\vec{f}$ is not globally invertible: $\vec{f}\left(\begin{bmatrix} -x \\ -y \end{bmatrix}\right) = \vec{f}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$

(2 to 1)

Local inverse ?

To check this : $D\vec{f} = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$

$\det D\vec{f} = 4(x^2+y^2) \geq 0$ & " $=0$ " $\Leftrightarrow (x,y)=(0,0)$

For $(x,y) \neq (0,0)$, IFT (Inverse Function Thm) $\Rightarrow$
$\vec{f}$ has a local inverse at (x,y) $(\neq (0,0))$

For instance, let $(x,y) = (1,-1)$ &
$\vec{g}(u,v)$ be a local inverse of $f(x,y)$
"near" $(x,y) = (1,-1)$

$\left(\text{where } \begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases} \right)$

$\vec{f}(1,-1) = (0,-2) \Rightarrow \vec{g}(0,-2) = (1,-1)$

$$D\vec{g}(0,-2) = \left(D\vec{f}(1,-1)\right)^{-1} = \left(\begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}_{(1,-1)} \right)^{-1}$$

$$= \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}^{-1} = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{(check!)}$$

<u>Explicit calculation of $\vec{g}(u,v)$</u> :

$$\begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$$

near $(x,y) = (1,-1) \Rightarrow x \neq 0 \Rightarrow y = \frac{v}{2x}$

$$\Rightarrow \quad u = x^2 - \left(\frac{v}{2x}\right)^2$$

$$\Rightarrow \quad 4x^4 - 4ux^2 - v^2 = 0$$

$$\Rightarrow \quad x^2 = \frac{4u \pm \sqrt{(-4u)^2 - 4\cdot4(-v^2)}}{8}$$

$$= \frac{u \pm \sqrt{u^2+v^2}}{2}$$

Put $(x,y) = (1,-1) \Rightarrow (u,v) = (0,-2)$

$$1^2 = \frac{0 \pm \sqrt{0^2+(-2)^2}}{2}$$

$\Rightarrow$ "$-$" should be rejected

$$\Rightarrow \quad x^2 = \frac{u+\sqrt{u^2+v^2}}{2}$$

$$\Rightarrow \quad x = \sqrt{\frac{u+\sqrt{u^2+v^2}}{2}}$$ ("$-$" rejected as x near 1)

& hence $y = \dfrac{2v}{x} = \dfrac{2\sqrt{2}\,v}{\sqrt{u+\sqrt{u^2+v^2}}}$

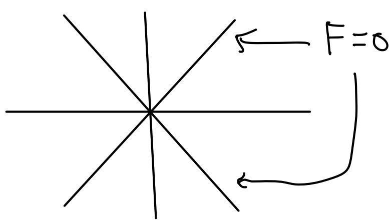
$$g(u,v) = \left(\sqrt{\frac{u+\sqrt{u^2+v^2}}{2}},\ \frac{2\sqrt{2}\,v}{\sqrt{u+\sqrt{u^2+v^2}}} \right) \quad \text{near } (0,-2)$$

Remark: In Implicit Function Thm & Inverse Function Thm, we need to check det. of Jacobian matrix (a submatrix) is nonzero. In case that the det $= 0$, we have No conclusion:

eg: Implicit Function Thm

$F(x,y) = x^2 - y^2 = 0$

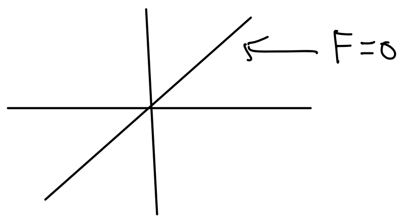
$$\boxed{\frac{\partial F}{\partial y} = -2y \Big|_{(0,0)} = 0}$$



y is not locally a function of x near (0,0)

$F(x,y) = x^3 - y^3 = 0$

$$\boxed{\frac{\partial F}{\partial y} = -3y^2 \Big|_{(0,0)} = 0}$$

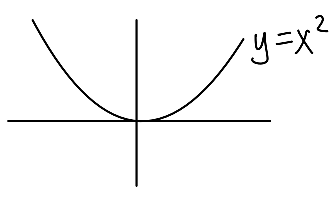


y is locally a function of x near (0,0)

Inverse function Thm

$f(x) = x^2$

$$\boxed{f'(0) = 0}$$

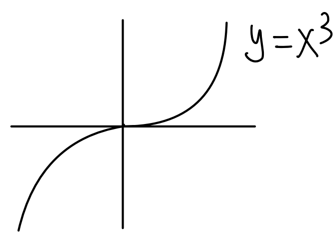


Not injective near $x=0$

$\Rightarrow$ no local inverse near $x=0$

$f(x) = x^3$

$$\boxed{f'(0) = 0}$$



$g(y) = y^{1/3}$ is a local inverse near $x=0$