(Cont'd)

(1) $\Rightarrow \lambda \neq 0$

then (2) $\Rightarrow$ $x + 2y = 0 \Rightarrow x = -2y$

Sub into (3) $\Rightarrow$ $(-2y)^2 + (-2y)y + y^2 = 9$

$\Rightarrow$ $y = \pm\sqrt{3}$ (Check!)

Hence $(x, y) = (-2\sqrt{3}, \sqrt{3}), (2\sqrt{3}, -\sqrt{3})$ are the critical points.

Comparing the values $2\sqrt{3} > -2\sqrt{3}$

$\Rightarrow$ max. value of x-coordinates is $2\sqrt{3}$

(at the point $(2\sqrt{3}, -\sqrt{3})$) ✕✕

<u>eg 2</u> Find the point(s) on the <u>hyperboloid</u>

$xy - yz - zx = 3$ (Check: it is really a hyperboloid)

↑ of 2 sheets

closest to the origin

<u>Soln</u> Let $f(x, y, z) = x^2 + y^2 + z^2$

$g(x, y, z) = xy - yz - zx$

Minimize $f$ under constraint $g = 3$

Consider $F(x, y, z, \lambda) = x^2 + y^2 + z^2 - \lambda(xy - yz - zx - 3)$

$$\begin{cases} 0 = \dfrac{\partial F}{\partial x} = 2x - \lambda(y-z) & \text{——— (1)} \\ 0 = \dfrac{\partial F}{\partial y} = 2y - \lambda(x-z) & \text{——— (2)} \\ 0 = \dfrac{\partial F}{\partial z} = 2z + \lambda(x+y) & \text{——— (3)} \\ 0 = \dfrac{\partial F}{\partial \lambda} = -(xy-yz-zx-3) & \text{——— (4)} \end{cases}$$

If $\lambda = 0$, then (1), (2) & (3) $\Rightarrow$ $x=y=z=0$

contradicting eqt. (4)

So $\lambda \neq 0$.

Then (1), (2) & (3) $\Rightarrow$
$$\begin{cases} y-z = \dfrac{2}{\lambda}x & \text{—— (5)} \\ x-z = \dfrac{2}{\lambda}y & \text{—— (6)} \\ x+y = -\dfrac{2}{\lambda}z & \text{—— (7)} \end{cases}$$

(5)−(6) $\Rightarrow$ $y-x = \frac{2}{\lambda}(x-y) \Rightarrow \left(1+\frac{2}{\lambda}\right)(x-y) = 0$ —— (8)

(7)−(6) $\Rightarrow$ $y+z = -\frac{2}{\lambda}(z+y) \Rightarrow \left(1+\frac{2}{\lambda}\right)(y+z) = 0$ —— (9)

If $1+\frac{2}{\lambda} = 0$, i.e. $\lambda = -2$,

then (5), (6), (7) $\Rightarrow$ $x+y-z=0$

$\Rightarrow$

$$\begin{aligned} 0 = (x+y-z)^2 &= x^2+y^2+z^2+2(xy-yz-xz) \\ &= x^2+y^2+z^2+6 \quad (\text{by } (4)) \end{aligned}$$

which is a contradiction.

$\therefore \quad 1+\frac{2}{\lambda} \neq 0$

Then (8) & (9) $\Rightarrow \quad x = y = -z$

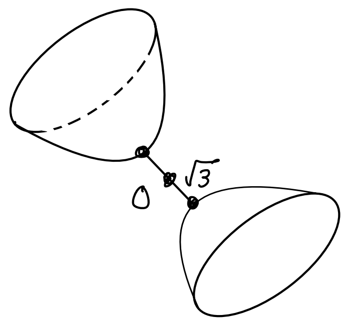
Sub. into (4) $\Rightarrow \quad 3x^2 = 3 \quad \Rightarrow \quad x = \pm 1$

$\therefore \quad (x, y, z) = \pm (1, 1, -1)$ ( & $\lambda = 1$, if you're interested)

$f(1,1,-1) = f(-1,-1,1) = 3$

$\Rightarrow$ closest points are $\pm(1,1,-1)$

with corresponding distance $= \sqrt{3}$



# Lagrange Multipliers with multiple Constraints

Let $\begin{cases} \bullet\ f, g_1, \cdots, g_k : \Omega \to \mathbb{R} \text{ be } C^1 \text{ functions}, \ (\Omega \subseteq \mathbb{R}^n, \text{ open}) \\ \bullet\ S = \{\vec{x} \in \Omega : g_i(\vec{x}) = c_i \text{ for } i = 1, \cdots, k\} \end{cases}$

Suppose $\begin{cases} \bullet\ \vec{a} \text{ is a local extremum of } f \text{ on } S \\ \bullet\ \vec{\nabla} g_1(\vec{a}), \cdots, \vec{\nabla} g_k(\vec{a}) \text{ are linearly independent vectors} \end{cases}$

Then

$$\begin{cases} \vec{\nabla} f(\vec{a}) = \sum_{i=1}^{k} \lambda_i \vec{\nabla} g_i(\vec{a}) \\ g_i(\vec{a}) = c_i, \quad i = 1, \cdots, k \end{cases}$$

for some Lagrange multipliers $\lambda_1, \cdots, \lambda_k \in \mathbb{R}$.

Same as 1 constraint,

Finding extrema of $f(\vec{x})$ with constraints $g_i(\vec{x}) = c_i$, $i = 1, \cdots, k$

$$\Updownarrow$$

Finding extrema of $F(\vec{x}, \lambda_1, \cdots, \lambda_k) = f(\vec{x}) - \sum_{i=1}^{k} \lambda_i (g_i(\vec{x}) - c_i)$

without constraint

(but more variables : adding $\lambda_i$ as new variables) eg1 Maximize $f(x,y,z) = x^2 + 2y - z^2$

on the line $L : \begin{cases} 2x - y = 0 \\ y + z = 0 \end{cases}$ in $\mathbb{R}^3$

(Given that maximum exists)

Soln Let $g_1(x,y,z) = 2x - y$

$g_2(x,y,z) = y + z$

Maximize $f$ subject to constraints $\begin{cases} g_1 = 0 \\ g_2 = 0 \end{cases}$

$\left[ \begin{array}{l} f \text{ is 2-degree poly,} \\ g_1, g_2 \text{ are degree 1 polynomials} \end{array} \Rightarrow f, g_1, g_2 \text{ are } C^1 \right]$

$\left.\begin{array}{l} \vec{\nabla} g_1 = (2 \quad -1 \quad 0) \\ \vec{\nabla} g_2 = (0 \quad 1 \quad 1) \end{array}\right\}$ are linearly independent (prove it!)

Consider

$$F(x,y,z,\lambda_1,\lambda_2) = x^2 + 2y - z^2 - \lambda_1(2x - y) - \lambda_2(y + z)$$

$$\begin{cases} 0 = \dfrac{\partial F}{\partial x} = 2x - 2\lambda_1 \\[2ex] 0 = \dfrac{\partial F}{\partial y} = 2 + \lambda_1 - \lambda_2 \\[2ex] 0 = \dfrac{\partial F}{\partial z} = -2z - \lambda_2 \\[2ex] 0 = \dfrac{\partial F}{\partial \lambda_1} = -(2x - y) \\[2ex] 0 = \dfrac{\partial F}{\partial \lambda_2} = -(y + z) \end{cases} \Leftrightarrow \begin{cases} x = \lambda_1 & \text{——(1)} \\[2ex] \lambda_2 = \lambda_1 + 2 & \text{——(2)} \\[2ex] \lambda_2 = -2z & \text{——(3)} \\[2ex] 2x = y & \text{——(4)} \\[2ex] y = -z & \text{——(5)} \end{cases}$$

(1) & (3) sub into (2)

$$-2z = x+2 \quad \text{——— (6)}$$

(4) & (5) $\Rightarrow$ $\quad 2x = y = -z \quad$ ——— (7)

sub into (6) $\quad 4x = x+2 \Rightarrow x = \frac{2}{3}$

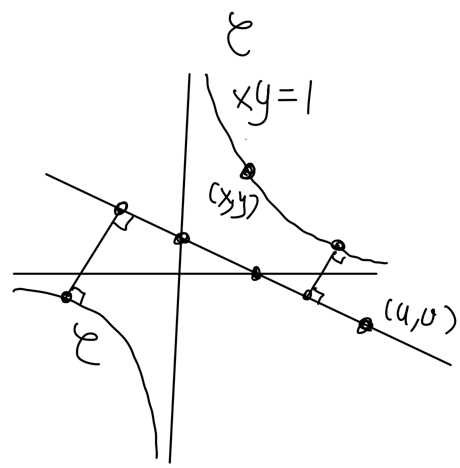
sub into (7) $\Rightarrow$ $\quad y = \frac{4}{3}, \ z = -\frac{4}{3}$

$\Rightarrow$ max occurs at $\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right)$

with value $f\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right) = \left(\frac{2}{3}\right)^2 + 2\left(\frac{4}{3}\right) - \left(\frac{4}{3}\right)^2$

(check!) $= \frac{4}{3}$ #

<u>eg2</u> Find the distance between the hyperbola $\mathcal{E}: xy = 1$ and the line $L: x + 4y = \frac{15}{8}$




<u>Soln</u>: Let

$$f(x,y,u,v) = (x-u)^2 + (y-v)^2$$

Minimize $f$ under constraints

$$g_1(x,y,u,v) = xy = 1$$

$$g_2(x,y,u,v) = u + 4v = \frac{15}{8}$$

$$\vec{\nabla} g_1 = [\, y \quad x \quad 0 \quad 0 \,]$$

$$\vec{\nabla} g_2 = [\, 0 \quad 0 \quad 1 \quad 4 \,]$$

$\vec{\nabla} g_1$ & $\vec{\nabla} g_2$ are linearly independent

$\iff (x, y) \neq (0,0)$ $\qquad$ ( Can you prove it ? )

Consider

$$F(x, y, u, v, \lambda_1, \lambda_2) = (x-u)^2 + (y-v)^2 - \lambda_1 (xy - 1) - \lambda_2 \left(u + 4v - \frac{15}{8}\right)$$

$$\left\{ \begin{aligned}
0 &= \frac{\partial F}{\partial x} = 2(x-u) - \lambda_1 y && \text{———— } (1) \\
0 &= \frac{\partial F}{\partial y} = 2(y-v) - \lambda_1 x && \text{———— } (2) \\
0 &= \frac{\partial F}{\partial u} = -2(x-u) - \lambda_2 && \text{———— } (3) \\
0 &= \frac{\partial F}{\partial v} = -2(y-v) - 4\lambda_2 && \text{———— } (4) \\
0 &= \frac{\partial F}{\partial \lambda_1} = -(xy - 1) && \text{———— } (5) \\
0 &= \frac{\partial F}{\partial \lambda_2} = -\left(u + 4v - \frac{15}{8}\right) && \text{———— } (6)
\end{aligned} \right.$$

<u>Case 1</u> If $\lambda_1 = 0$ or $\lambda_2 = 0$, then

$$x = u \;\&\; y = v$$

sub into (6) $\implies$ $x = \frac{15}{8} - 4y$

sub into (5) $\implies$ $\left(\frac{15}{8} - 4y\right) y = 1$

$4y^2 - \frac{15}{8} y + 1 = 0$ has no (real) solution

Case 2 $\lambda_1 \neq 0$ & $\lambda_2 \neq 0$.

Then (3) & (4) $\Rightarrow$

$$\left.\begin{array}{l} \dfrac{x-u}{y-v} = \dfrac{1}{4} \\ \\ \text{\& } (1) \,\&\, (2) \Rightarrow \\ \\ \dfrac{x-u}{y-v} = \dfrac{y}{x} \end{array}\right\} \Rightarrow \quad x = 4y$$

sub. into (5) $\quad (4y)y = 1 \Rightarrow y = \pm\frac{1}{2}$

$\therefore \ (x, y) = \pm(2, \frac{1}{2}) \qquad (\neq (0,0))$

Then for $(2, \frac{1}{2})$, $\dfrac{2-u}{\frac{1}{2}-v} = \dfrac{1}{4} \Rightarrow$

$$\left.\begin{array}{l} 4u - v = \dfrac{15}{2} \\ u + 4v = \dfrac{15}{8} \end{array}\right\}$$

together (6)

$\Rightarrow (u, v) = (\frac{15}{8}, 0)$ (Check!)

Similarly for $(-2, -\frac{1}{2})$, we have $(u, v) = (-\frac{225}{136}, \frac{15}{17})$ (Ex!)

Comparing the values $\quad f(2, \frac{1}{2}, \frac{15}{8}, 0) = \frac{17}{64} \ (= (\text{dist})^2)$ (check!)

$f(-2, -\frac{1}{2}, -\frac{225}{136}, \frac{15}{17}) = \cdots > \frac{17}{64}$

check

$\Rightarrow$ distance between $\mathcal{C}$ and $L = \frac{\sqrt{17}}{8}$ (Check) ※

# Implicit Function Theorem

**Recall**: Implicit differentiation

e.g. $\quad x^2+y^2+z^2=2$ and if $z=z(x,y)$,

then

$$\frac{\partial}{\partial x}(x^2+y^2+z^2)=0 \quad \Rightarrow \quad 2x+2z\frac{\partial z}{\partial x}=0$$

$$\frac{\partial}{\partial y}(x^2+y^2+z^2)=0 \quad \Rightarrow \quad 2y+2z\frac{\partial z}{\partial y}=0$$

If the point $(x,y,z)$ satisfies $z\neq 0$

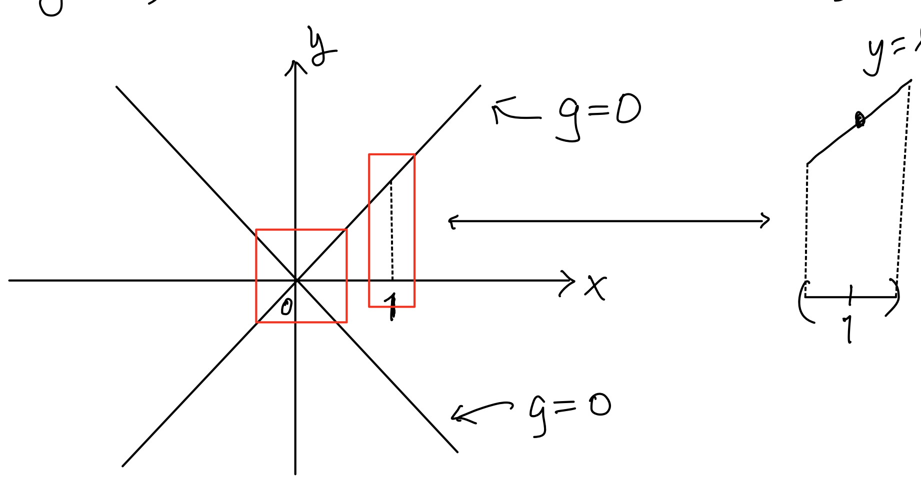
then we have $\quad \dfrac{\partial z}{\partial x}=-\dfrac{x}{z}$ & $\dfrac{\partial z}{\partial y}=-\dfrac{y}{z}$

**Question**: If a level set $g(x,y)=c$ (or more generally) is given, can we "solve" the constraint?

i.e. can we find $y=h(x)$ s.t. $g(x,h(x))=c$

or $\quad x=k(y)$ s.t. $g(k(y),y)=c$ ?

**eg1** $\quad g(x,y)=x^2-y^2=0 \quad (\Rightarrow \ x=\pm y)$




Yes, we can solve for $y=h(x)$ near $(x,y)=(1,1)$