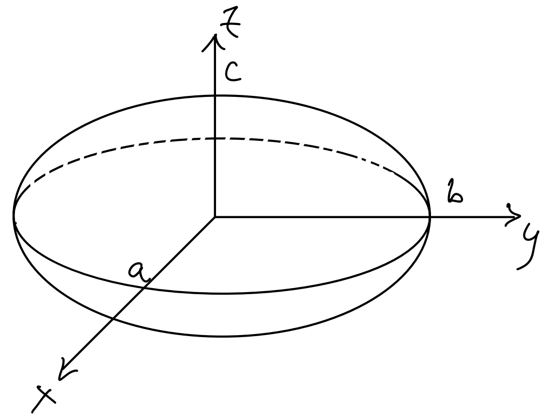
Quadratic Constraint for 3-variables

$$g(x,y,z) = Ax^2+By^2+Cz^2+2Pxy+2Qyz+2Rzx$$
$$+Dx+Ey+Fz+G$$

Same typical examples of $g=\text{const}$.

eg 1 $\quad \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$

Ellipsoid



eg2 $\quad \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$ Hyperboloid of 1 sheet

Up to scaling of each variables, graph looks like the graph of

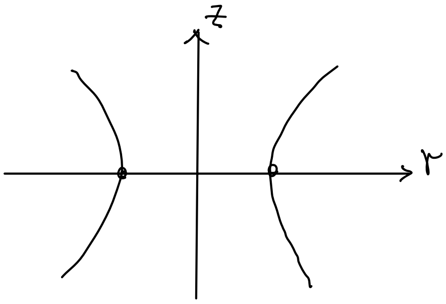
$$x^2+y^2-z^2=1$$

Using polar condinates on xy-plane,

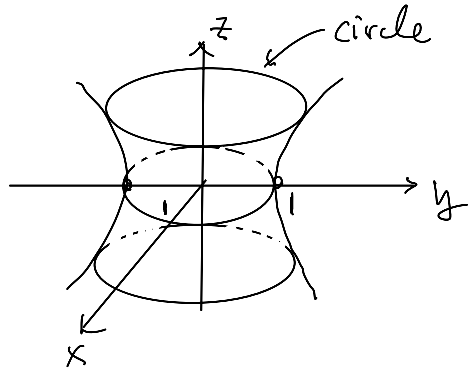
$$\Rightarrow x^2+y^2=r^2$$

$\therefore$ the constraint can be written as

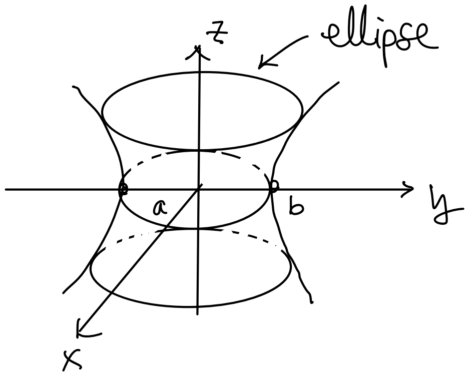
$$r^2-z^2=1$$



same for all directions of θ



$\boxed{\text{Hyperboloid of 1 sheet}}$



scaling back to

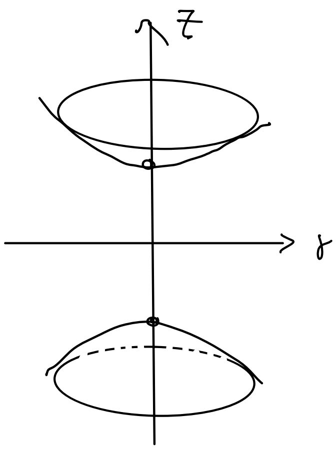
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

eg3 $\quad \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=-1 \qquad$ Hyperboloid of 2 sheets

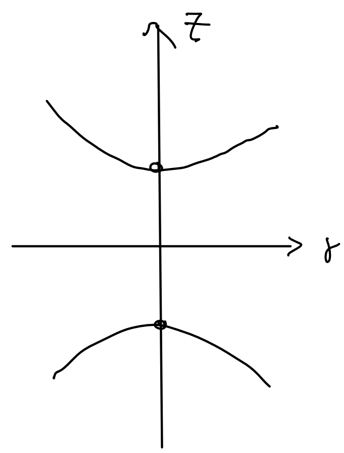
Similarly, after scaling, looks like

$$x^2+y^2-z^2=-1 \iff \text{in polar} \quad r^2-z^2=-1$$

$$\iff \quad z^2-r^2=1$$



same for each direction of θ



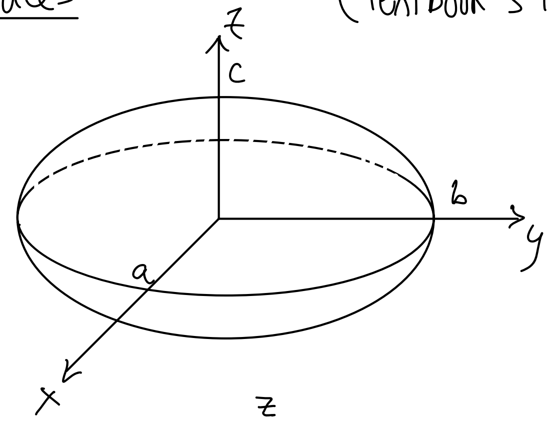
$\boxed{\text{Hyperboloid of 2 sheets}}$

In summary, we have

Graphs of Standard Quadratic Surfaces (Textbook §12.6)

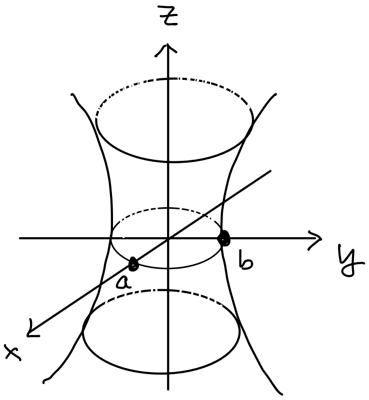
Ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$



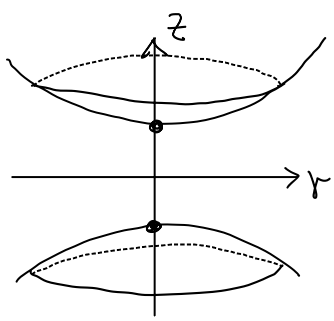
Hyperboloid of 1 sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$



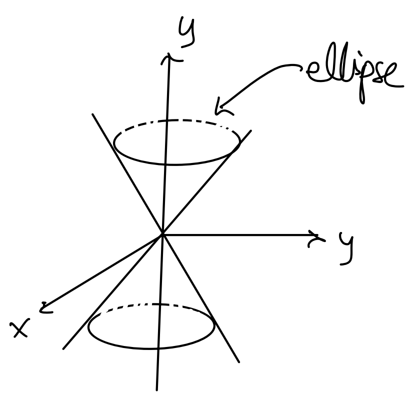
Hyperboloid of 2 sheets

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$$



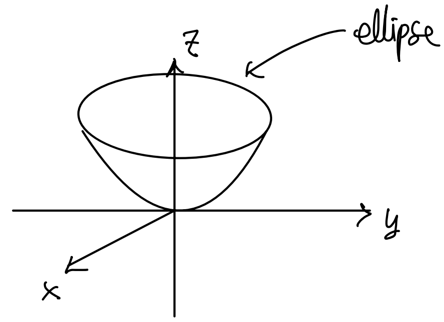
Elliptic Cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$



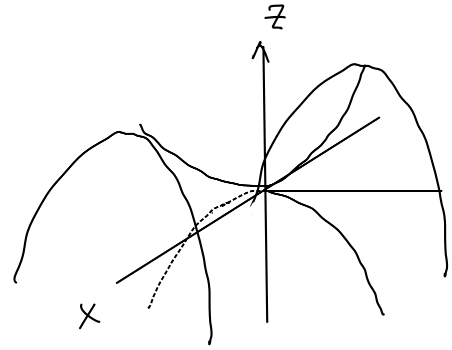
Elliptic Paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$



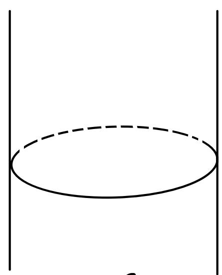
Hyperbolic Paraboloid

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

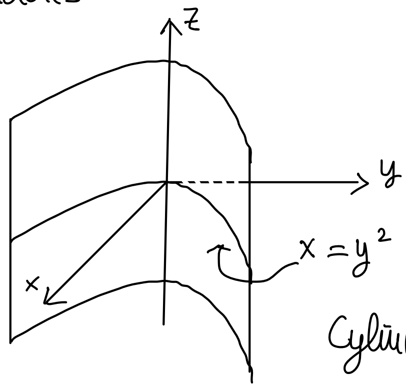


Degenerate to cylinders over conic sections

eg: Equation involve NO z variables



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ cylinder of ellipse



and etc.

$x = y^2$

Cylinder of parabola

Other degenerate cases

eg $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$ & $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1$

> **Fact** Any quadratic constraint $g(x,y,z) = c$ can be transformed to one of the standard forms by a change of coordinates.

Remark: As in 2-variables, only ellipsoid is closed and bounded

Further examples

eg1 Find the point on the ellipse

$$x^2+xy+y^2=9 \qquad \text{(Check: it is really a ellipse!)}$$

with maximum x-coordinate.

Soln: Let $f(x,y)=x$ & $g(x,y)=x^2+xy+y^2$

Maximize f under the constraint $g=9$

Consider $F(x,y,\lambda)=x-\lambda(x^2+xy+y^2-9)$

$$\begin{cases} 0=\dfrac{\partial F}{\partial x}=1-\lambda(2x+y) \\ 0=\dfrac{\partial F}{\partial y}=-\lambda(x+2y) \\ 0=\dfrac{\partial F}{\partial \lambda}=-(x^2+xy+y^2-9) \end{cases} \Leftrightarrow \begin{cases} \lambda(2x+y)=1 \quad \text{---(1)} \\ \lambda(x+2y)=0 \quad \text{---(2)} \\ x^2+xy+y^2=9 \quad \text{---(3)} \end{cases}$$

(To be cont'd)