Lagrange Multipliers

(A method for *finding extrema under constraints*)

eg1 In previous example of finding global max/min of $f(x,y) = x^2+2y^2-x+3$ for $x^2+y^2 \le 1$, one need to find (in step 2) the max/min values of f on the boundary $x^2+y^2=1$.

In otherwords, finding global max/min (on $x^2+y^2=1$)

of $$f(x,y) = x^2+2y^2-x+3$$

under constraint $$g(x,y) = x^2+y^2 = 1$$

Another typical example :

eg2 Find the point on the parabola $x^2=4y$ closest to $(1,2)$.

i.e. Find (global) minimum of

$$f(x,y) = (x-1)^2+(y-2)^2$$

(equivalent to, but easier than $\sqrt{(x-1)^2+(y-2)^2}$)

under constraint

$$g(x,y) = x^2-4y = 0 .$$

Remark : In both examples, constraints are expressed as level set. $g=c$ for some constant c.

Thm (Lagrange Multipliers)

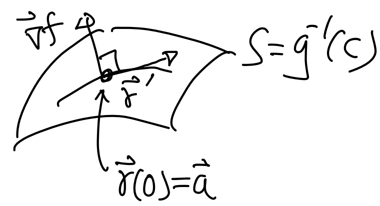
Let
- $f, g : \Omega \to \mathbb{R}$ be C^1 functions, ($\Omega \subset \mathbb{R}^n$ open)
- $S = g^{-1}(c) = \{x \in \Omega : g(x) = c\}$ be a level set of g

Suppose
- $\vec{a} \in S$ is a local extremum of f restricted to S (i.e. under the constraint $g = c$)
- $\vec{\nabla} g(\vec{a}) \neq \vec{0}$

Then
- $\vec{\nabla} f(\vec{a}) = \lambda \vec{\nabla} g(\vec{a})$ for some $\lambda \in \mathbb{R}$
- $g(\vec{a}) = c$

where λ is called a Lagrange Multiplier

(Pf: Omitted) Idea: $f(\vec{\gamma}(t))$ has an local extreme at $\vec{a}$ $\Rightarrow 0 = \frac{d}{dt}\Big|_{t=0} f(\vec{\gamma}(t))$



$\Rightarrow \quad 0 = \vec{\nabla} f(\vec{a}) \cdot \vec{\gamma}\,'(0)$ (by Chain rule)

Since it is true for all curves on S passing thro. $\vec{a}$,

$\Rightarrow \quad \vec{\nabla} f(\vec{a})$ perpendicular to $S = g^{-1}(c)$ at $\vec{a}$

$\Rightarrow \quad \vec{\nabla} f(\vec{a})$ is in the direction (or negative) of $\vec{\nabla} g(\vec{a})$.

i.e. $\vec{\nabla} f(\vec{a}) = \lambda \vec{\nabla} g(\vec{a})$ for some $\lambda \in \mathbb{R}$ #

Reduction to unconstrained problem (By Lagrange Multiplier)

Finding extrema of $f(\vec{x})$ with constraint $g(\vec{x})=c$

$$\Updownarrow$$

Finding extrema of $F(\vec{x},\lambda) = f(\vec{x}) - \lambda(g(\vec{x})-c)$

without constraint

(but more variables : adding λ as a new variable)

Idea: $F(\vec{x},\lambda) = F(x_1,\cdots,x_n,\lambda)$ is $n+1$ variables

$$\underset{\mathbb{R}^{n+1}}{\vec{0}} = \underset{n+1\text{-variable}}{\vec{\nabla}} F = \left(\frac{\partial F}{\partial x_1}, \cdots, \frac{\partial F}{\partial x_n}, \frac{\partial F}{\partial \lambda}\right)$$

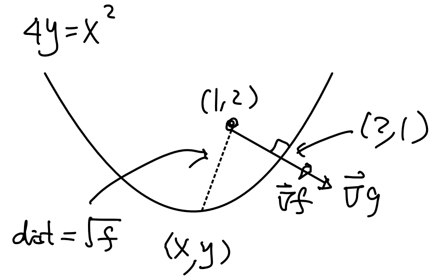
$$\begin{cases} 0 = \dfrac{\partial F}{\partial x_i} = \dfrac{\partial}{\partial x_i}\left(f - \lambda(g-c)\right) & \forall\, i=1,\cdots,n \\ \quad = \dfrac{\partial f}{\partial x_i} - \lambda \dfrac{\partial g}{\partial x_i} \\ 0 = \dfrac{\partial F}{\partial \lambda} = \dfrac{\partial}{\partial \lambda}\left(f - \lambda(g-c)\right) \\ \quad = -(g-c) \end{cases}$$

$$\Leftrightarrow \begin{cases} \vec{\nabla} f = \lambda \vec{\nabla} g \\ g = c \end{cases}$$

n-variable $\left(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_n}\right)$

eg2 (cont'd) minimize $f(x,y) = (x-1)^2+(y-2)^2$

under constraint $g(x,y) = x^2-4y=0$



Soln: Consider

$$F(x,y,\lambda) = f(x,y) - \lambda(g(x,y)-c)$$

$$= (x-1)^2+(y-2)^2 - \lambda(x^2-4y)$$

$$\begin{cases} 0 = \dfrac{\partial F}{\partial x} = 2(x-1) - 2\lambda x & \text{—— (1)} \\ 0 = \dfrac{\partial F}{\partial y} = 2(y-2) + 4\lambda & \text{—— (2)} \\ 0 = \dfrac{\partial F}{\partial \lambda} = -(x^2-4y) & \text{—— (3)} \end{cases}$$

(2) $\Rightarrow$ $2\lambda = 2-y$

Put in (1) $\Rightarrow$ $0 = 2(x-1) - x(2-y)$

$\Rightarrow$ $y = \dfrac{2}{x}$

Put in (3) $\Rightarrow$ $x^2 - \dfrac{8}{x} = 0 \Rightarrow x=2$

& hence $y=1$

$\therefore$ $(2,1)$ is the only critical point

$\Rightarrow$ f has a minimum at $(2,1) \in g^{-1}(0)$

with value $f(2,1) = (2-1)^2+(1-2)^2 = 2$

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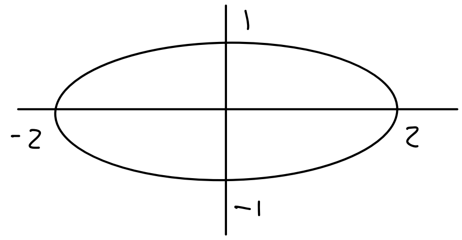
eg2 Maximize $f(x,y) = xy^2$ on the ellipse $x^2+4y^2=4$

Soln : $\begin{cases} f(x,y)=xy^2 \\ g(x,y) = x^2+4y^2 \end{cases}$



& hence consider

$$F(x,y,\lambda) = f(x,y) - \lambda(g(x,y)-4)$$
$$= xy^2 - \lambda(x^2+4y^2-4)$$

$$\begin{cases} 0 = \frac{\partial F}{\partial x} = y^2 - 2\lambda x \\ 0 = \frac{\partial F}{\partial y} = 2xy - 8\lambda y \\ 0 = \frac{\partial F}{\partial \lambda} = -(x^2+4y^2-4) \end{cases}$$

By "simple" calculation, we have

$$(x,y) = (\pm 2, 0) \quad \text{or} \quad \left(\pm\sqrt{\tfrac{4}{3}}, \pm\sqrt{\tfrac{2}{3}}\right)$$

$$\left(= \begin{matrix}(2,0) \\ (-2,0)\end{matrix} \quad \& \quad \begin{matrix} \left(\frac{2}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right) & \left(-\frac{2}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right) \\ \left(-\frac{2}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right) & \left(-\frac{2}{\sqrt{3}}, -\sqrt{\frac{2}{3}}\right) \end{matrix} \right)$$

are all critical points of the problem.

Comparing values of f at all these 6 critical points:

$$f(\pm 2, 0) = 0$$

$$f\left(\frac{2}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right) = \frac{2}{\sqrt{3}} \cdot \frac{2}{3} = \frac{4}{3\sqrt{3}} \longleftarrow \text{max}$$

$$f\left(-\frac{2}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right) = -\frac{2}{\sqrt{3}} \cdot \frac{2}{3} = -\frac{4}{3\sqrt{3}} \longleftarrow \text{min}$$

$\therefore$ For $f(x,y)$ on $g(x,y)=4$, the

global max value $= \frac{4}{3\sqrt{3}}$ at $\left(\frac{2}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right)$

global min value $= \frac{-4}{3\sqrt{3}}$ at $\left(\frac{-2}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right)$

$\#$

eg 1 (cont'd) Using Lagrange multiplier, find global max/min of

$$f(x,y) = x^2+2y^2-x+3 \quad \text{on} \quad x^2+y^2=1$$

(Step 2 of the original global max/min problem on $x^2+y^2\le 1$)

Soln: Let $f(x,y) = x^2+2y^2-x+3$

$g(x,y) = x^2+y^2$

& $F(x,y,\lambda) = x^2+2y^2-x+3-\lambda(x^2+y^2-1)$

$$\begin{cases} 0 = \frac{\partial F}{\partial x} = 2x-1-2\lambda x \\ 0 = \frac{\partial F}{\partial y} = 4y-2\lambda y \\ 0 = \frac{\partial F}{\partial \lambda} = -(x^2+y^2-1) \end{cases}$$

Simple calculation $\Rightarrow$ $(x,y) = (\pm 1, 0)$, $\left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$

Comparing values

$$f\left(-\tfrac{1}{2}, \pm\tfrac{\sqrt{3}}{2}\right) = \tfrac{21}{4} \leftarrow \text{max} \quad (\text{on } x^2+y^2=1)$$

$$f(1,0) = 3 \leftarrow \text{min} \quad (\text{on } x^2+y^2=1)$$

$$f(-1,0) = 5$$

$\therefore$ Global max of f on $x^2+y^2=1$ $= \frac{21}{4}$ at $\left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$

Global min of f on $x^2+y^2=1$ $= 3$ at $(1, 0)$

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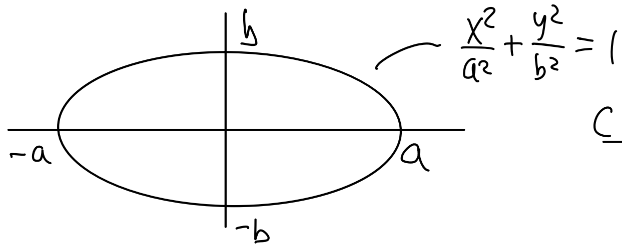
Classification of Quadratic Constraints

2-variables : $\quad g(x,y) = Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F$

(Conic Section)

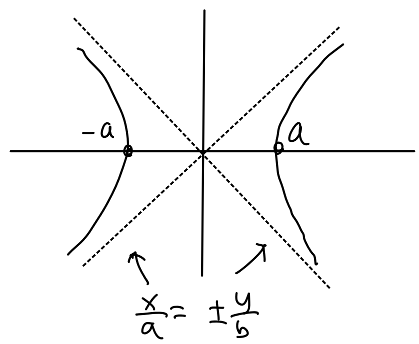
Typical examples for level curve $g(x,y) = c$

(i) $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, \quad a, b > 0$ $\quad$ Ellipse (circle if $a=b$)



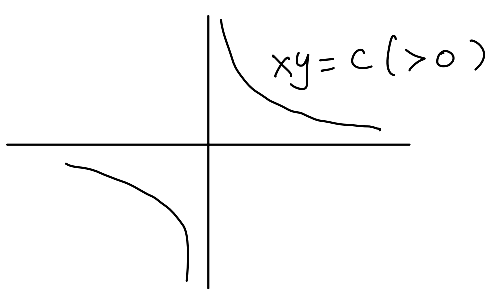
closed & bounded

(ii) $\quad \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1, \quad a, b > 0$ $\quad$ Hyperbola



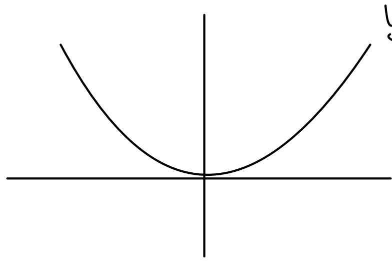
unbounded

($xy = c$, $c \neq 0$, is also a hyperbola)



(iii') $y = ax^2$, $a \neq 0$ <u>Parabola</u>

(only 1 quadratic term)



<u>unbounded</u>

(iv) <u>Degenerate Cases</u> $(a>0, b>0)$

- $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 0 \longrightarrow$ a point $(0,0)$

- $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = -1 \longrightarrow$ empty set

- $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 0 \longrightarrow \dfrac{x}{a} = \pm\dfrac{y}{b}$ a pair of intersecting lines

  $(xy = 0)$

- $x^2 = c \longrightarrow x = \pm\sqrt{c}$
  - a pair of parallel lines if $c>0$
  - a "double" line if $c=0$
  - empty set if $c<0$

<u>Fact</u>: By a change of coordinates, any <u>quadratic constraint</u> $g(x,y)=c$ can be transformed to one of the form above.

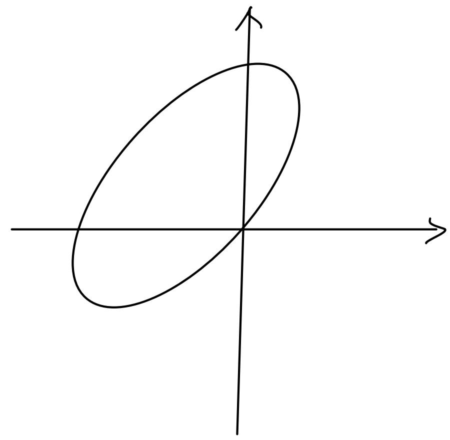
(<u>Proof</u>: Omitted)

Hence level sets of quadratic constraints are ellipse, hyperbola, parabola, & degenerated cases.

eg $17x^2 - 12xy + 8y^2 + 16\sqrt{5}x - 8\sqrt{5}y = 0$

$$\Leftrightarrow \quad \frac{u^2}{1^2} + \frac{v^2}{2^2} = 1$$

where $\begin{cases} u = \dfrac{2x-y}{\sqrt{5}} + 1, \\ v = \dfrac{x+2y}{\sqrt{5}} \end{cases}$



Remark: Ellipse is closed and bounded $\Rightarrow$ Any continuous $f(x,y)$ restricted to an ellipse has global max & min.

Not the case for hyperbola & parabola.