eg 1 $f(x,y) = 3x^2 - 10xy + 3y^2 + 2x + 2y + 3$

Find and classify critical points of f.

Soln: (f polynomial, always C^2)

$$\vec{\nabla} f = [\,6x - 10y + 2 \quad -10x + 6y + 2\,]$$

$$Hf = \begin{bmatrix} 6 & -10 \\ -10 & 6 \end{bmatrix} \qquad \text{(a constant matrix)}$$

Critical point:

$$\vec{0} = \vec{\nabla} f \iff \begin{cases} 6x - 10y + 2 = 0 \\ -10x + 6y + 2 = 0 \end{cases} \overset{\text{check}}{\iff} (x,y) = \left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

$$\det Hf = f_{xx} f_{yy} - f_{xy}^2 = 6 \cdot 6 - (-10)^2 = -64 < 0$$

$\Rightarrow \left(\tfrac{1}{2}, \tfrac{1}{2}\right)$ is a saddle point (by 2^{nd} derivative test)

(No need to check $f_{xx} = 6 > 0$)

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eg 2: $f(x,y) = 3x - x^3 - 3xy^2$

Find and classify critical points of f

Soln (f polynomial, always C^2)

$$\vec{\nabla} f = [3-3x^2-3y^2, \quad -6xy]$$

$$Hf = \begin{bmatrix} -6x & -6y \\ -6y & -6x \end{bmatrix}$$

Critical points :

$$\vec{0} = \vec{\nabla} f \iff \begin{cases} 3-3x^2-3y^2 = 0 \\ -6xy = 0 \end{cases}$$

$$\overset{\text{check}}{\iff} (x,y) = (0,1), (0,-1), (1,0), (-1,0)$$

(4 critical points)

Critical point	Hf	$f_{xx}f_{yy}-f_{xy}^2 = \det Hf$	f_{xx}	classification
$(0,1)$	$\begin{bmatrix} 0 & -6 \\ -6 & 0 \end{bmatrix}$	$-36<0$	No need	saddle
$(0,-1)$	$\begin{bmatrix} 0 & 6 \\ 6 & 0 \end{bmatrix}$	$-36<0$	No need	saddle
$(1,0)$	$\begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix}$	$36>0$	$-6<0$	local max
$(-1,0)$	$\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$	$36>0$	$6>0$	local min

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eg3 (Inconclusive from 2^{nd} derivative test)

$f(x,y) = x^2+y^4$, $g(x,y) = x^2-y^4$, $h(x,y) = -x^2-y^4$

(at (0,0): local min , saddle , local max)

Soln

$\vec{\nabla} f = [2x \ \ 4y^3]$ $\quad \vec{\nabla} g = [2x \ \ -4y^3]$ $\quad \vec{\nabla} h = [-2x \ \ -4y^3]$

$(\vec{\nabla} f(0,0) = [0,0] = \vec{\nabla} g(0,0) = \vec{\nabla} h(0,0))$

$$Hf = \begin{bmatrix} 2 & 0 \\ 0 & 12y^2 \end{bmatrix} \quad Hg = \begin{bmatrix} 2 & 0 \\ 0 & -12y^2 \end{bmatrix} \quad Hh = \begin{bmatrix} -2 & 0 \\ 0 & -12y^2 \end{bmatrix}$$

$$Hf(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad Hg = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad Hh = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\det Hf(0,0) = \det Hg(0,0) = \det Hh(0,0) = 0 .$$

$\therefore$ 2^{nd} derivative test is inclusive.

Higher dimension example

eg: $q(x,y,z) = xy + yz + zx$

has definite sign for $(x,y,z) \neq (0,0,0)$?

Answer: No

Soln: Trick: $q(x,y,z) = \frac{1}{4}(x+y)^2 - \frac{1}{4}(x-y)^2 + z(x+y)$

Let $u = \frac{x+y}{2}$, $v = \frac{x-y}{2}$, then

$$\begin{aligned} q &= u^2 - v^2 + 2uz \\ &= (u^2 + 2uz + z^2) - v^2 - z^2 \\ &= (u+z)^2 - v^2 - z^2 \\ &= \frac{1}{4}(x+y+2z)^2 - \frac{1}{4}(x-y)^2 - z^2 \end{aligned}$$

- On the plane $x+y+2z=0$ (i.e. $z = -\frac{x+y}{2}$)

$$q = q(x,y,-\tfrac{x+y}{2}) = -\frac{1}{4}(x-y)^2 - \frac{1}{4}(x+y)^2$$
$$< 0 \quad \text{for } (x,y,-\tfrac{x+y}{2}) \neq \vec{0}$$

- Along the line $\begin{cases} x-y=0 \\ z=0 \end{cases} \Rightarrow \begin{cases} x=y \\ z=0 \end{cases}$

$$q = q(x,x,0) = \frac{1}{4}(x+x+0)^2 - 0^2 - 0^2 = x^2 > 0$$
$$(\forall x \neq 0, \text{ i.e. } \forall (x,x,0) \neq \vec{0})$$

(Together $\Rightarrow$ $(0,0,0)$ is a saddle point)

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Second Derivative Test for general n

Recall f is $C^2 \Rightarrow$ (by Clairaut's / mixed derivative Thm)

$$Hf(\vec{a}) = \left[f_{x_i x_j} \right]_{i,j=1,\dots,n} \text{ is symmetric}$$

Theory of Linear Algebra $\Rightarrow$ Hf is diagonizable

i.e. $\exists$ orthogonal $n \times n$ matrix P (i.e $P^T P = Id$) s.t.

$$P^T Hf(\vec{a}) P = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

where λ_i, $i = 1, \dots, n$, are eigenvalues of $Hf(\vec{a})$.

$$\Rightarrow \boxed{Hf(\vec{a}) \text{ is } \begin{cases} \text{positive definite} \iff \text{all } \lambda_i > 0 \\ \text{negative definite} \iff \text{all } \lambda_i < 0 \\ \text{indefinite} \iff \text{some } \lambda_i > 0, \text{ some } \lambda_j < 0 \text{ (all } \neq 0) \end{cases}}$$

Another way to check is consider determinants of submatrix

For each $1 \le k \le n$,

consider submatrix H_k given by the upper left $k \times k$ entries.

$$\left[\begin{array}{ccc|ccc} f_{x_1 x_1} & \cdots & f_{x_1 x_k} & \cdots & f_{x_1 x_n} \\ \vdots & & \vdots & & \vdots \\ f_{x_k x_1} & \cdots & f_{x_k x_k} & \cdots & f_{x_k x_n} \\ \hline \vdots & & \vdots & & \vdots \\ f_{x_n x_1} & \cdots & f_{x_n x_k} & \cdots & f_{x_n x_n} \end{array}\right]$$

Then

$$\boxed{\begin{aligned} &Hf(\vec{a}) \text{ is positive definite} \iff \det H_k > 0, \ \forall k = 1, \cdots, n \\ &Hf(\vec{a}) \text{ is negative definite} \iff \det H_k \begin{cases} < 0, & k \text{ odd} \\ > 0, & k \text{ even} \end{cases} \end{aligned}}$$

egs (1) $n=2$ $\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$ has $H_1 = [f_{xx}]$

$H_2 = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$

$\Rightarrow \quad \det H_1 = f_{xx}$

$\det H_2 = f_{xx} f_{yy} - f_{xy}^2$ (Same result as before)

(2) Diagonal matrix $\begin{bmatrix} \lambda_1 & & & & 0 \\ & \ddots & & & \\ & & \lambda_k & & \\ & 0 & & \ddots & \\ & & & & \lambda_n \end{bmatrix} \Rightarrow H_k = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_k \end{bmatrix}$

$\Rightarrow \det H_k = \lambda_1 \cdots \lambda_k$.