Application to local max/min

If f is C^2, and $\vec{a}$ is a critical point of f.

Then $\vec{\nabla} f(\vec{a}) = \vec{0}$

$$\Rightarrow \quad f(\vec{x}) \approx P_2(\vec{x}) = f(\vec{a}) + \cancelto{\vec{0}}{\vec{\nabla} f(a)}(\vec{x}-\vec{a}) + \frac{1}{2}(\vec{x}-\vec{a})^T Hf(a)(\vec{x}-\vec{a})$$

$$f(\vec{x}) - f(\vec{a}) \approx \frac{1}{2}(\vec{x}-\vec{a})^T Hf(a)(\vec{x}-\vec{a})$$

$\therefore$ If $(\vec{x}-\vec{a})^T Hf(a)(\vec{x}-\vec{a}) < 0 \quad \forall \, \vec{x}$ near $\vec{a}$

then $f(\vec{x}) < f(\vec{a}) \quad \forall \, \vec{x}$ near $\vec{a}$

$\Rightarrow$ $\vec{a}$ is a <u>local max</u>

If $(\vec{x}-\vec{a})^T Hf(a)(\vec{x}-\vec{a}) > 0 \quad \forall \, \vec{x}$ near $\vec{a}$

then $f(\vec{x}) > f(\vec{a}) \quad \forall \, \vec{x}$ near $\vec{a}$

$\Rightarrow$ $\vec{a}$ is a <u>local min</u>

So we need to study when is a sym matrix H satisfies

$$\vec{v}^T H \vec{v} > 0 \quad \forall \text{ vector } \vec{v} \neq \vec{0}$$

and

$$\vec{v}^T H \vec{v} < 0 \quad \forall \text{ vector } \vec{v} \neq \vec{0}$$

Hence we make the following

Def: Let H be a symmetric $n \times n$ matrix.
Then H is said to be

(1) positive definite if $\vec{x}^T H \vec{x} > 0$
for all column vector $\vec{x} \in \mathbb{R}^n \setminus \{\vec{0}\}$

(2) negative definite if $\vec{x}^T H \vec{x} < 0$
for all column vector $\vec{x} \in \mathbb{R}^n \setminus \{\vec{0}\}$

(3) indefinite if $\exists$ column vectors $\vec{x}, \vec{y} \in \mathbb{R}^n \setminus \{\vec{0}\}$
such that $\vec{x}^T H \vec{x} > 0$ and $\vec{y}^T H \vec{y} < 0$

Remark: These are not all possibilities : $\exists$ sym. matrix which is not positive definite, negative definite, nor indefinite.

Then the discussion above implies

Thm (Second Derivative Test)

Let
- $f : \Omega \to \mathbb{R}$ be C^2, $\Omega \subseteq \mathbb{R}^n$, open
- $\vec{a} \in \Omega$ such that $\vec{\nabla} f(\vec{a}) = \vec{0}$

Then

$$Hf(\vec{a}) \text{ is } \begin{cases} \text{positive definite} \Rightarrow \vec{a} \text{ is a local min} \\ \text{negative definite} \Rightarrow \vec{a} \text{ is a local max} \\ \text{indefinite} \Rightarrow \vec{a} \text{ is a saddle point} \end{cases}$$

Remark: A critical point which is neither local max nor local min is called a saddle point.

In particular for 2-variable, $\vec{v}^T H \vec{v}$ is of the form

$$g(x,y) = [x \ y]\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + 2bxy + cy^2$$

eg (1) $[x,y]\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x^2 + 4y^2 > 0, \ \forall \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \setminus \{\vec{0}\}$

$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ is positive definite.

(2) $[x,y]\begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -x^2 - 4y^2 < 0, \ \forall \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \setminus \{\vec{0}\}$

$\therefore \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}$ is negative definite.

(3) $[x,y]\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -x^2 + 4y^2$

If $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $[x,y]\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -0^2 + 4(1)^2 = 4 > 0$

If $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $[x,y]\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -(1)^2 + 4(0)^2 = -1 < 0$

$\therefore \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$ is indefinite.

(4) $[x,y]\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x^2 \geq 0, \ \forall \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \setminus \{\vec{0}\}$

But $[0 \ 1]\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \Rightarrow$ not positive definite

$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is not positive definite, negative definite, nor indefinite.

(5) $[x, y]\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x^2 + 4xy + 5y^2$

$$= x^2 + 4xy + 4y^2 + y^2$$

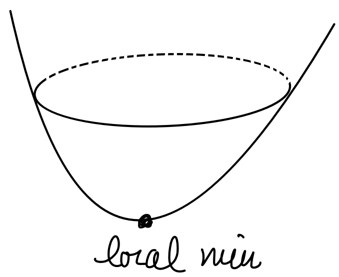
$$= (x+2y)^2 + y^2$$

$$> 0 \qquad \forall \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \setminus \{\vec{0}\}$$

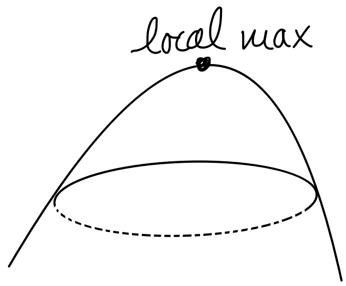
$\Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ is positive definite.

<u>Geometrically</u> (locally near the <u>critical point</u>)

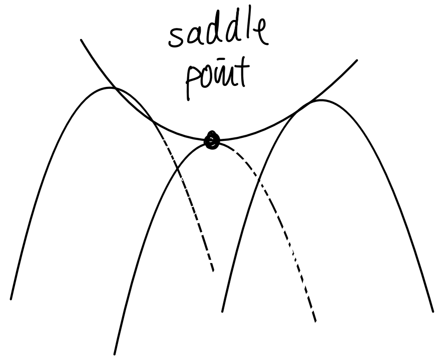
(1) $Hf(\vec{a})$ positive definite



(2) $Hf(\vec{a})$ negative definite



(3) $Hf(\vec{a})$ is indefinite



Thm Let $H = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$.

Then

$$H \text{ is } \begin{cases} \underline{\text{positive definite}} \iff \det H = ac - b^2 > 0, \ \& \ a > 0 \\ \underline{\text{negative definite}} \iff \det H = ac - b^2 > 0, \ \& \ a < 0 \\ \underline{\text{indefinite}} \iff \det H = ac - b^2 < 0 \end{cases}$$

Pf: Using completing square

If $a \neq 0$, $[x \ y] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + 2bxy + cy^2$

$$= a(x^2 + 2\frac{b}{a}xy) + cy^2 = a(x + \frac{b}{a}y)^2 + \frac{(ac - b^2)}{a}y^2$$

$$\left(= a\left[(x + \frac{b}{a}y)^2 + \frac{ac - b^2}{a^2}y^2 \right] \right)$$

(1) $\det H = ac - b^2 > 0 \iff$

$[x \ y] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ has the same sign as a.

$\therefore$ The 1st 2 statements are proved.

(2) $\det H = ac - b^2 < 0 \iff$

2 terms have different sign

$\iff \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is indefinite.

If $a = 0 \Rightarrow \det H = ac - b^2 = -b^2 \leq 0$

$[x \ y] \begin{bmatrix} 0 & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2bxy + cy^2$

If $b=0$, then $[x\ y]\begin{bmatrix}0 & b\\ b & c\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = cy^2$ not positive, negative definite nor indefinite

If $b\neq 0$, then $2bxy+cy^2$

$$= 2by\left(x+\frac{c}{2b}y\right)$$

$$= \frac{b}{2}\left[\left(y+\left(x+\frac{c}{2b}y\right)\right)^2 - \left(y-\left(x+\frac{c}{2b}y\right)\right)^2\right]$$

(using $4uv = (u+v)^2-(u-v)^2$)

$\Rightarrow$ also indefinite

Still in the case of $\det H = -b^2 < 0 \Leftrightarrow$ indefinite ※

eg1 $q(x,y) = 2xy = [x\ y]\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$

$\det H = -1 < 0 \Rightarrow$ indefinite

$(a=0)$ $\quad\left(q(x,y) = \frac{1}{2}(x+y)^2 - \frac{1}{2}(x-y)^2\right)$

eg2: $q(x,y) = 17x^2-12xy+8y^2 = [x\ y]\begin{bmatrix}17 & -6\\ -6 & 8\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$

$$\left.\begin{array}{l}\det\begin{bmatrix}17 & -6\\ -6 & 8\end{bmatrix} = 17\cdot 8-(-6)^2 = 100>0,\\[2ex] a = 17 > 0\end{array}\right\}\Rightarrow \text{positive definite}$$

$$\left(q(x,y) = 17\left(x-\frac{6}{17}y\right)^2 + \frac{100}{17}y^2\right)$$

Then for 2-variable, the 2nd derivative test is

Thm (Second Derivative Test for 2-variables)

Let $\begin{cases} \bullet\ f : \Omega \to \mathbb{R} \text{ be } C^2,\ \Omega \subseteq \mathbb{R}^2, \text{ open} \\ \bullet\ (a,b) \in \Omega \text{ such that } \vec{\nabla} f(a,b) = \vec{0} \end{cases}$

Then

(1) $f_{xx} f_{yy} - f_{xy}^2 > 0$ & $f_{xx} > 0$ at $(a,b) \Rightarrow (a,b)$ is a local min

(2) $f_{xx} f_{yy} - f_{xy}^2 > 0$ & $f_{xx} < 0$ at $(a,b) \Rightarrow (a,b)$ is a local max

(3) $f_{xx} f_{yy} - f_{xy}^2 < 0$ at $(a,b) \Rightarrow (a,b)$ is a saddle point

(4) $f_{xx} f_{yy} - f_{xy}^2 = 0$ at $(a,b) \Rightarrow$ inconclusive.

Remark $\quad f_{xx} f_{yy} - f_{xy}^2 = \det Hf \quad$ (for 2-variables)

eg (1) $\quad f(x,y) = x^3 + y^2$

$$f_{xx} = 6x \quad f_{xy} = 0, \quad f_{yy} = 2$$

$$Hf = \begin{bmatrix} 6x & 0 \\ 0 & 2 \end{bmatrix} \qquad \det Hf = 12x$$

$$\vec{\nabla} f = \vec{0} \Leftrightarrow (3x^2, 2y) = (0,0) \Leftrightarrow (x,y) = (0,0)$$

$$\det Hf(0,0) = 0.$$

$f(t,0) > 0$, $f(-t,0) < 0$ for all $t > 0$ small

$\Rightarrow (0,0)$ is a saddle point

(2) $g(x,y) = x^4 + y^4$ & $h(x,y) = -x^4 - y^4$

Clearly $(0,0)$ is a critical point of both g & h

$$Hg = \begin{bmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{bmatrix}, \quad Hh = \begin{bmatrix} -12x^2 & 0 \\ 0 & -12y^2 \end{bmatrix}$$

$\Rightarrow \det Hg(0,0) = 0 = \det Hh(0,0)$

$(0,0)$ is a minimum point of g, but

$(0,0)$ is a maximum point of h

(1) & (2) $\Rightarrow$ critical point (a,b), can be local max, local min, a saddle in the case when $\det Hf(a,b) = 0$.