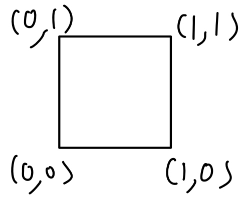
(Cont'd) $\quad f(x,y) = 6xy - 4x^3 - 3y^2$

Interior critical point $(\frac{1}{2}, \frac{1}{2})$ with value $f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4}$.

<u>Boundary</u>



(1) $\{x=0,\ 0 \le y \le 1\}$: $\quad f(0,y) = -3y^2$

$$\Rightarrow f(0,1) = -3 \le f(0,y) \le 0 = f(0,0)$$

(2) $\{y=0,\ 0 \le x \le 1\}$: $\quad f(x,0) = -4x^3$

$$\Rightarrow -4 = f(1,0) \le f(x,0) \le 0 = f(0,0)$$

(3) $\{x=1,\ 0 \le y \le 1\}$: $\quad f(1,y) = 6y - 4 - 3y^2 = -3(y-1)^2 - 1$

$$f(1,0) = -4 \le f(1,y) \le -1 = f(1,1) \qquad \left(\begin{array}{c} -1 \le y-1 \le 0 \\ \Rightarrow (y-1)^2 \le 1 \end{array}\right)$$

(4) $\{y=1,\ 0 \le x \le 1\}$: $\quad f(x,1) = 6x - 4x^3 - 3$

$$0 = \frac{d}{dx} f(x,1) = 6 - 12x^2 \Rightarrow x = \frac{1}{\sqrt{2}} \quad \left(-\frac{1}{\sqrt{2}} \text{ rejected as } 0 \le x \le 1\right)$$

$$f(\tfrac{1}{\sqrt{2}}, 1) = \frac{6}{\sqrt{2}} - 4\left(\frac{1}{\sqrt{2}}\right)^3 - 3 = -(3 - 2\sqrt{2}) < 0 \qquad \text{(check!)}$$

$$f(0,1) = -3, \quad f(1,1) = -1 \quad (< -(3-2\sqrt{2}))$$

$$\Rightarrow \quad -3 = f(0,1) \le f(x,1) \le f(\tfrac{1}{\sqrt{2}}, 1) = -(3 - 2\sqrt{2})$$

<u>Compare values</u>

Global max. pt. $(\frac{1}{2}, \frac{1}{2})$ with value $\frac{1}{4}$

Global min. pt. $(1,0)$ with value -4 ※

Matrix form for 2^{nd} order Taylor Polynomial

Def: Let $f: \Omega \to \mathbb{R}$ be C^2 ($\Omega \subseteq \mathbb{R}^n$, open).
Then the Hessian matrix of f at $\vec{a} \in \Omega$ is

$$Hf(\vec{a}) = \begin{bmatrix} f_{x_1x_1}(\vec{a}) & \cdots & f_{x_1x_n}(\vec{a}) \\ \vdots & & \vdots \\ f_{x_nx_1}(\vec{a}) & \cdots & f_{x_nx_n}(\vec{a}) \end{bmatrix}$$

Remarks (1) $Hf(\vec{a})$ is $n \times n$ symmetric (by Clairaut's Thm)

(2) In Textbook, Hessian of f = $\det(Hf(\vec{a}))$

So we emphasize our definition is a matrix (More common in advanced level math)

eg: $f(x,y)$ at $(0,0)$

$$Hf(0,0) = \begin{bmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{yx}(0,0) & f_{yy}(0,0) \end{bmatrix} \qquad (f_{xy} = f_{yx})$$

$$[x, y] \begin{bmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{yx}(0,0) & f_{yy}(0,0) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \underbrace{f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2}$$

2^{nd} order term in Taylor polynomials (up to a factor $\frac{1}{2!}$)

2^{nd} order Taylor polynomial of f at $\vec{a}$ in matrix form

$$\boxed{P_2(\vec{x}) = f(\vec{a}) + \vec{\nabla} f(\vec{a})(\vec{x}-\vec{a}) + \frac{1}{2}(\vec{x}-\vec{a})^T Hf(\vec{a})(\vec{x}-\vec{a})}$$

where $\vec{\nabla} f(\vec{a})$ regarded as row vector $[f_{x_1}(\vec{a}) \cdots f_{x_n}(\vec{a})]$,

$\vec{x}-\vec{a}$ regarded as column vector $\begin{bmatrix} x_1-a_1 \\ \vdots \\ x_n-a_n \end{bmatrix}$

& $(\vec{x}-\vec{a})^T$ is the transpose $[x_1-a_1 \cdots x_n-a_n]$ (row vector)

eg $g(x,y) = \frac{\ln x}{1-y}$. Find $P_2(x,y)$ at $(1,0)$ using matrix form.

Soln: $g(1,0) = 0$

$$\vec{\nabla} g = \left[\frac{\partial g}{\partial x} \quad \frac{\partial g}{\partial y}\right] = \left[\frac{1}{x(1-y)} \quad \frac{\ln x}{(1-y)^2}\right]$$

$$Hg = \begin{bmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{bmatrix} = \begin{bmatrix} -\frac{1}{x^2(1-y)} & \frac{1}{x(1-y)^2} \\ \frac{1}{x(1-y)^2} & \frac{\ln x}{(1-y)^3} \end{bmatrix}$$

$$\Rightarrow \vec{\nabla} g(1,0) = [1, 0], \quad Hg(1,0) = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\therefore P_2(x,y) = g(1,0) + \vec{\nabla} g(1,0)\begin{bmatrix} x-1 \\ y \end{bmatrix} + \frac{1}{2}[x-1 \quad y]\, Hg(1,0)\begin{bmatrix} x-1 \\ y \end{bmatrix}$$

$$= 0 + [1 \quad 0]\begin{bmatrix} x-1 \\ y \end{bmatrix} + \frac{1}{2}[x-1 \quad y]\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x-1 \\ y \end{bmatrix}$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + (x-1)y \quad \text{(Check!)}$$

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