

Finding Extrema (Maximum & Minimum)

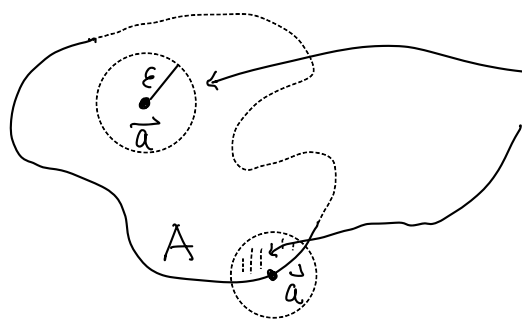
Def: Let $f: A \rightarrow \mathbb{R}$, $A \subset \mathbb{R}^n$ (may not be open)
| $\vec{a} \in A$

(1) f is said to have a global (absolute) maximum at \vec{a}
if $f(\vec{a}) \geq f(\vec{x}) \quad \forall \vec{x} \in A$

(2) f is said to have a local (relative) maximum at \vec{a}
if $f(\vec{a}) \geq f(\vec{x}) \quad \forall \vec{x} \in A$ "near" \vec{a}
(i.e. $\exists \epsilon > 0$ s.t. $f(\vec{a}) \geq f(\vec{x}), \forall \vec{x} \in A \cap B_\epsilon(\vec{a})$)

(3) Similar definitions for global (absolute) minimum and
local (relative) minimum by changing the inequality
to $f(\vec{a}) \leq f(\vec{x})$.

global max at \vec{a}
means $f(\vec{a}) \geq f(\vec{x})$
on the whole A



local max at \vec{a}
means $f(\vec{a}) \geq f(\vec{x})$
only holds inside
the ball.

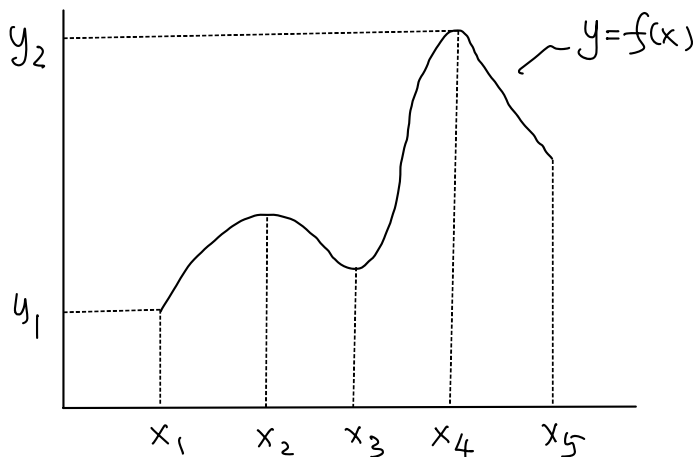
Remark: Global Extremum (max/min) is also a
local extremum.

eg1 $f: [x_1, x_5] \rightarrow \mathbb{R}$

Global $\left\{ \begin{array}{l} \text{max} = x_4 \\ \text{min} = x_1 \end{array} \right.$

Local $\left\{ \begin{array}{l} \text{max} = x_2, x_4 \\ \text{min} = x_1, x_3, x_5 \end{array} \right.$

max value = y_2
min = y_1



eg2 (NOT every function has global max/min)

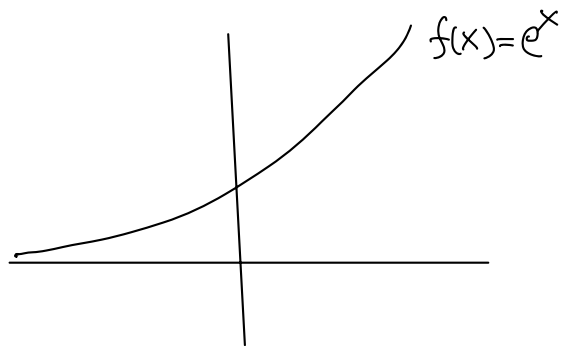
(i) $f(x) = e^x$ on \mathbb{R} (Domain unbounded)

No global min: $\lim_{x \rightarrow -\infty} f(x) = 0$

($\Rightarrow \forall a \in \mathbb{R}, f(a) = e^a > 0$. The limit $\Rightarrow \exists x \in \mathbb{R}$
s.t. $f(x) < f(a)$)

No global max: $\lim_{x \rightarrow +\infty} f(x) = +\infty$

($\Rightarrow \forall a \in \mathbb{R}, \exists x \in \mathbb{R}$ s.t. $f(x) > f(a) = e^a$)



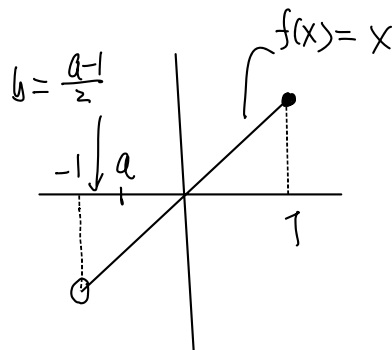
(ii) $f(x) = x$ on $(-1, 1]$ (Domain not closed)

Has global max at $x=1$ (Easy)

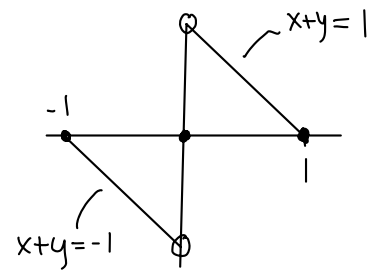
(with max value = $f(1) = 1$),

but no global min, $\forall a \in (-1, 1]$,

$\exists b = \frac{a-1}{2} \in (-1, 1]$ s.t. $f(b) < f(a)$



$$(iii) f = \begin{cases} 1-x, & 0 < x \leq 1 \\ 0, & x=0 \\ -1-x, & -1 \leq x < 0 \end{cases} \text{ (discontinuous)}$$



No global max and global min

(Ex! Similar argument as in (ii))

Extreme Value Thm (EVT)

Let $f: A \subseteq \mathbb{R}^n$ be closed and bounded,
 $f: A \rightarrow \mathbb{R}$ be continuous

Then f has global max and min.

(Proof: Omitted)

Remarks: (1) "Compact" = closed and bounded

(2) The Thm is a sufficient, but not a necessary condition.

Def: Let $f: A \rightarrow \mathbb{R}$, $A \subseteq \mathbb{R}^n$ (not necessary open)
 $\vec{a} \in \text{Int}(A)$

Then \vec{a} is called a critical point of f if

either (1) $\vec{\nabla} f(\vec{a})$ DNE (does not exist)

or (2) $\vec{\nabla} f(\vec{a}) = \vec{0}$

(i.e. either " $\frac{\partial f}{\partial x_i}(\vec{a})$ DNE for some $i=1, \dots, n$ " or " $\frac{\partial f}{\partial x_i}(\vec{a}) = 0$ for all $i=1, \dots, n$ ")

Thm (First Derivative Test)

Suppose $f: A \rightarrow \mathbb{R}$ ($A \subset \mathbb{R}^n$) attains a local extremum at $\vec{a} \in \text{Int}(A)$, then \vec{a} is a critical point of f .

Pf: Suppose f has a local extremum at $\vec{a} \in \text{Int}(A)$.

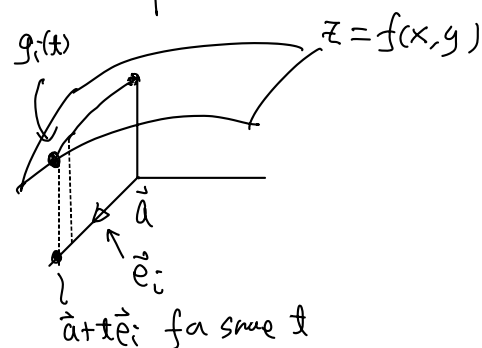
If $\vec{\nabla}f(\vec{a})$ DNE, then \vec{a} is a critical point.

If $\vec{\nabla}f(\vec{a})$ exists, then

$$\frac{\partial f}{\partial x_i}(\vec{a}), \forall i=1, \dots, n$$

$\forall i$, consider the 1-variable function

$$g_i(t) = f(\vec{a} + t\vec{e}_i) \quad (i=1, \dots, n)$$



$$\vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{th component}$$

Then $g_i(0) = f(\vec{a}) \stackrel{\leq}{\geq} f(\vec{a} + t\vec{e}_i)$ for $|t|$ small enough
st $\vec{a} + t\vec{e}_i$ is "near" \vec{a}

$\Rightarrow t=0$ is a local max/min

$$1\text{-variable theory} \Rightarrow 0 = g_i'(0) = \left. \frac{d}{dt} \right|_{t=0} f(\vec{a} + t\vec{e}_i)$$

$$= \vec{\nabla}f(\vec{a}) \cdot \vec{e}_i$$

$$= \frac{\partial f}{\partial x_i}(\vec{a}) \quad (\forall i)$$

$$\Rightarrow \vec{\nabla}f(\vec{a}) = \vec{0}$$

✱

Strategy for finding Extrema

$$f: A \rightarrow \mathbb{R}$$

(1) Find critical points of f in $\text{Int}(A)$.

(2) Study f on boundary ∂A :

Find max/min of f on ∂A .

(3) Compare values of f at points found in steps (1) & (2).

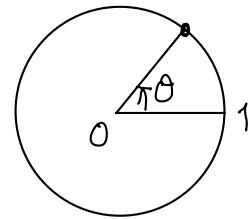
eg 1 Find global max/min of f on A

$$f(x,y) = x^2 + 2y^2 - x + 3 \quad \text{for } x^2 + y^2 \leq 1 \quad (A = \{x^2 + y^2 \leq 1\})$$

closed & bdd

Soln : Step 1

$$\begin{aligned} \text{Critical points in } \text{Int}(A) &= \text{Int}\{x^2 + y^2 \leq 1\} \\ &= \{x^2 + y^2 < 1\} \end{aligned}$$



closed unit disk

(f is a polynomial $\Rightarrow \vec{\nabla} f$ always exist)

$$\vec{\nabla} f = (2x - 1, 4y)$$

$$\vec{\nabla} f = \vec{0} \Leftrightarrow \begin{cases} 2x - 1 = 0 \\ 4y = 0 \end{cases} \Leftrightarrow (x,y) = (\frac{1}{2}, 0)$$

the only critical pt. in $\text{Int}(A)$.

$$a \quad f\left(\frac{1}{2}, 0\right) = \left(\frac{1}{2}\right)^2 + 0 - \frac{1}{2} + 3 = \frac{11}{4} \quad (\text{check!})$$

Step 2 Study f on $\partial\{x^2+y^2 \leq 1\} = \{x^2+y^2=1\}$

parametrize the boundary $\{x^2+y^2=1\}$ by angle θ :

$$\begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases} \quad \theta \in [0, 2\pi]$$

Therefore on $\partial\{x^2+y^2 \leq 1\}$,

$$\begin{aligned} f(\theta) &= f(\cos\theta, \sin\theta) = \cos^2\theta + 2\sin^2\theta - \cos\theta + 3 \\ &= -\cos^2\theta - \cos\theta + 5 \\ &= -\left(\cos\theta + \frac{1}{2}\right)^2 + \frac{21}{4} \quad (\text{check!}) \end{aligned}$$

$$\text{max value of } f \text{ on } \partial A = \frac{21}{4} \quad (\text{at } \cos\theta = -\frac{1}{2} \Rightarrow (x, y) = \left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right))$$

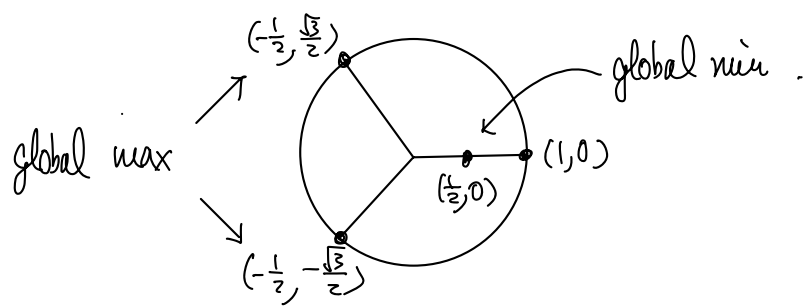
$$\begin{aligned} \text{min value of } f \text{ on } \partial A &= -\left(1 + \frac{1}{2}\right)^2 + \frac{21}{4} = 3 \quad (\text{check!}) \\ &(\text{at } \cos\theta = 1 \Rightarrow (x, y) = (1, 0)) \end{aligned}$$

Step 3 Compare values of f at points from steps (1) & (2)

$$f\left(\frac{1}{2}, 0\right) = \frac{11}{4} \quad (\text{value of critical pt. in Int}(A))$$

$$f\left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right) = \frac{21}{4} \quad (\text{max value of } f \text{ on } \partial A)$$

$$f(1, 0) = 3 \quad (\text{min value of } f \text{ on } \partial A)$$



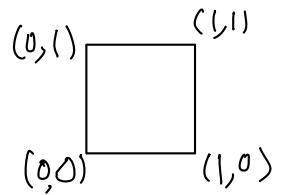
$$\Rightarrow \begin{cases} \text{max value} = \frac{21}{4} & \text{at the global max points } \left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right) \\ \text{min value} = \frac{11}{4} & \text{at the global min point } \left(\frac{1}{2}, 0\right) \end{cases}$$

✘

eg2 Find global max & min and values of

$$f(x,y) = 6xy - 4x^3 - 3y^2$$

on the set $\{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1\}$



Solu Step1: Interior extremum

(f is a polynomial, $\vec{\nabla}f$ always exist)

$$\begin{cases} 0 = \frac{\partial f}{\partial x} = 6y - 12x^2 \\ 0 = \frac{\partial f}{\partial y} = 6x - 6y \end{cases}$$

$$\Rightarrow y = 2x^2 \quad \& \quad x = y$$

$$\Rightarrow x = 2x^2$$

$$\Rightarrow x = 0 \text{ or } x = \frac{1}{2}$$

Hence $(x,y) = (0,0)$ & $(x,y) = \left(\frac{1}{2}, \frac{1}{2}\right)$

↑
on boundary

↑
interior.

values

$$f(0,0) = 0 \quad (\text{boundary})$$

$$f\left(\frac{1}{2}, \frac{1}{2}\right) = 6 \cdot \frac{1}{2} \cdot \frac{1}{2} - 4\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^2$$

$$= \frac{1}{4} \quad (\text{check!})$$

(To be cont'd)