# Application of Chain Rule

## Level Set

Recall : $f : \Omega \to \mathbb{R}$ , $\Omega \subseteq \mathbb{R}^n$

$$\vec{x} \longrightarrow c = f(\vec{x}), \quad c \in \mathbb{R}$$

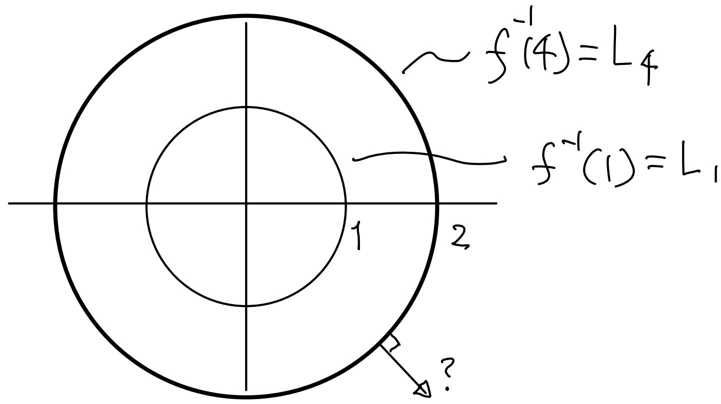
$L_c = f^{-1}(c) = \{\vec{x} \in \Omega : f(\vec{x}) = c\}$.

Level set of $f$ (at level $c$).

eg : $f(x,y) = x^2 + y^2$

$f^{-1}(1) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$

$f^{-1}(4) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 4\}$




**Thm** Let
- $f : \Omega \to \mathbb{R}$ ($\Omega \subset \mathbb{R}^n$, open)
- $c \in \mathbb{R}$
- $\vec{a} \in S = f^{-1}(c)$ ($S$ = a level set of $f$)

Suppose
- $f$ is differentiable at $\vec{a}$,
- $\vec{\nabla} f(\vec{a}) \neq \vec{0}$

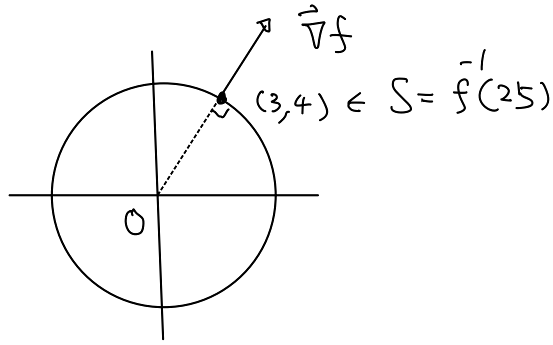
Then $\vec{\nabla} f(\vec{a}) \perp S$ at $\vec{a}$

eg 1 $\quad f(x,y) = x^2+y^2$

$\vec{\nabla} f = (2x, 2y)$

At $(3,4) \in S = f^{-1}(25)$

(i.e. level $c = 25$)



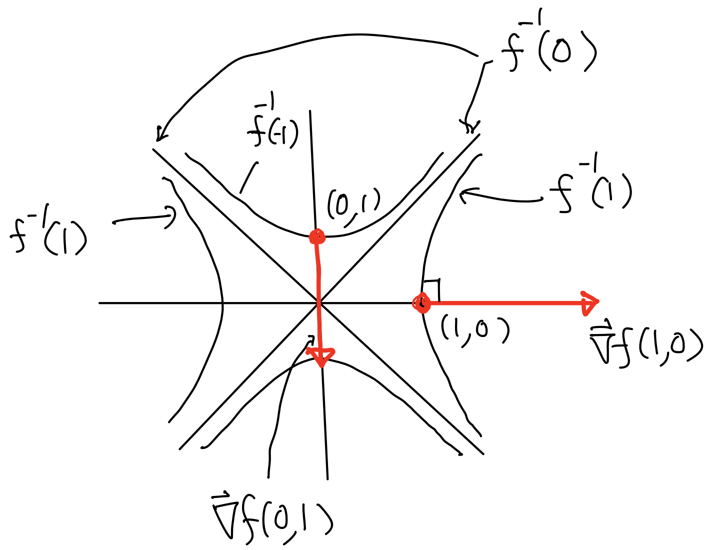

$\vec{\nabla} f(3,4) = (6,8) \perp S = f^{-1}(25)$ at the point $(3,4)$

eg 2 $\quad f(x,y) = x^2 - y^2$

$\vec{\nabla} f = (2x, -2y)$

$\vec{\nabla} f(1,0) = (2,0)$

$\vec{\nabla} f(0,1) = (0,-2)$

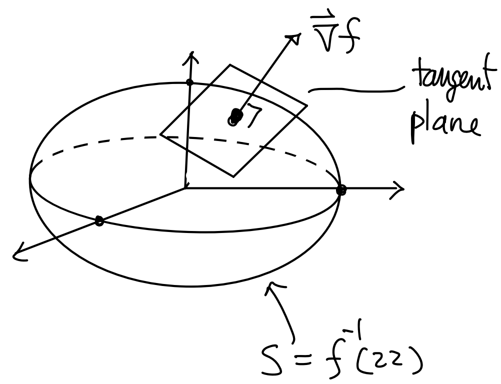



(Try other points)

(What happen at $(0,0) \in f^{-1}(0)$ ?)

eg 3 $\quad S: x^2 + 4y^2 + 9z^2 = 22$ (Ellipsoid)

Find equation of tangent plane of $S$ at the point $(3,1,1)$




(Check: $(3,1,1)$ is on $S$)

Soln: Let $f(x,y,z) = x^2 + 4y^2 + 9z^2$

Then $S = f^{-1}(22)$

$\vec{\nabla} f = (2x, 8y, 18z)$

$\Rightarrow \quad \vec{\nabla} f(3,1,1) = (6,8,18) \perp S \quad$ at $(3,1,1)$

i.e. $\vec{\nabla} f(3,1,1)$ is a normal to the tangent plane at $(3,1,1)$

$$\Rightarrow \quad \vec{\nabla} f(3,1,1) \cdot (x-3, y-1, z-1) = 0$$

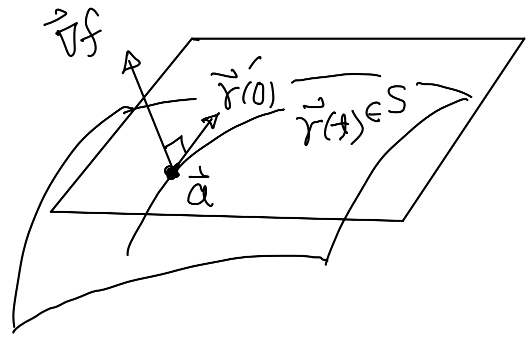
is the equation of the tangent plane.

$$\therefore \quad 6(x-3) + 8(y-1) + 18(z-1) = 0$$

or $\quad 3x + 4y + 9z = 22$

is the required equation of the tangent plane at $(3,1,1)$. ※

Proof of the Thm: $(\vec{\nabla} f \perp S)$




Let $\vec{\gamma}(t)$ be a curve on $S$ passing through the point $\vec{a}$ such that $\vec{\gamma}(0) = \vec{a}$

Then $f(\vec{\gamma}(t)) = c, \ \forall t \quad$ (because $\vec{\gamma}(t) \in S = f^{-1}(c)$)

Chain rule $\Rightarrow$

$$0 = \left.\frac{d}{dt}\right|_{t=0} \left( f(\vec{\gamma}(t)) \right) = \left. \vec{\nabla} f(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t) \right|_{t=0}$$

$$= \vec{\nabla} f(\vec{a}) \cdot \underbrace{\vec{\gamma}'(0)}$$

tangent vector of $\vec{\gamma}(t)$ which is also a tangent vector to $S$.

$\vec{\nabla} f(\vec{a}) \perp$ all curves on $S$ at $\vec{a}$

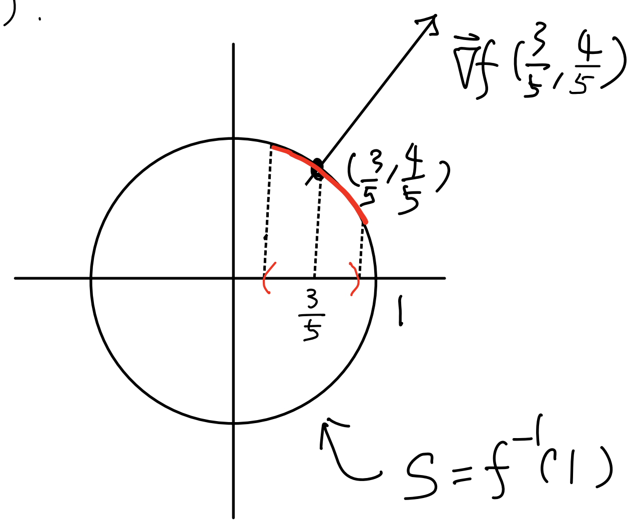
$\Rightarrow \vec{\nabla} f(\vec{a}) \perp S$ at $\vec{a}$ ※

# Another Application of Chain Rule:

## Implicit Differentiation

eg 1 $\quad C: x^2+y^2=1 \quad$ ( $y$ can be solved in term of $x$ (for most $x$))

Find $\frac{dy}{dx}$ at $\left(\frac{3}{5}, \frac{4}{5}\right)$.




Soln: $\quad x^2 + (y(x))^2 = 1$

near $\quad (x,y) = \left(\frac{3}{5}, \frac{4}{5}\right)$

$$\Rightarrow \quad \frac{d}{dx}\left(x^2 + (y(x))^2\right) = 1$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y} \quad (\text{provided } y \neq 0)$$

At the point $\left(\frac{3}{5}, \frac{4}{5}\right) \Rightarrow \frac{dy}{dx} = -\frac{3}{4}$

Remark: One cannot solve $y$ as a function of $x$ near the points $(1,0)$ and $(-1,0)$ which correspond to "$y=0$".

eg 2 Consider $S: x^3+z^2+ye^{xz}+z\cos y=0$

Given that $z$ can be regarded as a function $z=z(x,y)$ of (independent) variables $x,y$ locally near the point $(0,0,0)$. (Clearly $(0,0,0)\in S$)

Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ at $(0,0,0)$

Soln: $\frac{\partial}{\partial x}(x^3+z^2+ye^{xz}+z\cos y)=0$

$$\Rightarrow \quad 3x^2+2z\frac{\partial z}{\partial x}+ye^{xz}\frac{\partial}{\partial x}(xz)+\frac{\partial z}{\partial x}\cos y=0$$

$$\Rightarrow \quad 3x^2+2z\frac{\partial z}{\partial x}+ye^{xz}\left(z+x\frac{\partial z}{\partial x}\right)+\frac{\partial z}{\partial x}\cos y=0$$

$$\Rightarrow \quad (3x^2+yze^{xz})+(2z+xye^{xz}+\cos y)\frac{\partial z}{\partial x}=0$$

$$\Rightarrow \quad \frac{\partial z}{\partial x}=-\frac{3x^2+yze^{xz}}{2z+xye^{xz}+\cos y}$$

(provided $2z+xye^{xz}+\cos y\neq 0$)

$$\Rightarrow \quad \frac{\partial z}{\partial x}(0,0)=0$$

Similarly, $\frac{\partial}{\partial y}(x^3+z^2+ye^{xz}+z\cos y)=0$

(check!) $\Rightarrow$

$$\frac{\partial z}{\partial y}=\frac{z\sin y-e^{xz}}{2z+xye^{xz}+\cos y}$$

(provided $2z+xye^{xz}+\cos y\neq 0$)

$$\Rightarrow \quad \frac{\partial z}{\partial y}(0,0)=-1 \quad ※$$