$\boxed{\begin{array}{l}\underline{\text{Thm}}\ \text{Suppose } f \text{ is } \underline{\text{differentiable}} \text{ at } \vec{a}. \\ \quad \text{Let } \vec{u} \text{ be a } \underline{\text{unit}} \text{ vector in } \mathbb{R}^n, \text{ then} \\ \qquad D_{\vec{u}} f(\vec{a}) = \vec{\nabla} f(\vec{a}) \cdot \vec{u}\end{array}}$

<u>eg</u>: Let $f(x,y) = \sin^{-1}\left(\frac{x}{y}\right)$.

Find the rate of change of f at $(1, \sqrt{2})$ in the direction of $\vec{v} = (1,-1)$ (not necessary unit).

<u>Remark</u>: $\vec{v} \neq \vec{0} \in \mathbb{R}^n$, not necessary unit, then the <u>direction of $\vec{v}$</u> is $\frac{\vec{v}}{\|\vec{v}\|}$ (a unit vector).

<u>Soln</u>: Let $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{1^2+(-1)^2}}(1,-1) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \sin^{-1}\left(\frac{x}{y}\right) = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{1}{y}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \sin^{-1}\left(\frac{x}{y}\right) = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \frac{\partial}{\partial y}\left(\frac{x}{y}\right) = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{-x}{y^2}$$

[Note, $f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are continuous "near" $(1,\sqrt{2}) \Rightarrow f$ is C^1 near $(1,\sqrt{2})$]

f is differentiable at $(1, \sqrt{2})$

$$\xRightarrow{\text{Thm}} \quad D_{\vec{u}} f(1,\sqrt{2}) = \vec{\nabla} f(1,\sqrt{2}) \cdot \vec{u}$$

$$= \left(\frac{\partial f}{\partial x}(1,\sqrt{2}), \frac{\partial f}{\partial y}(1,\sqrt{2}) \right) \cdot \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right)$$

$$= \cdots = \frac{1}{\sqrt{2}} + \frac{1}{2} \quad \text{(Check!)}$$

Pf: (<u>Differentiable $\Rightarrow D_{\vec{u}} f(\vec{a}) = \vec{\nabla} f(\vec{a}) \cdot \vec{u}$</u>)

Let $L(\vec{x})$ be the linearization of $f(\vec{x})$ at $\vec{a}$

& $f(\vec{x}) = L(\vec{x}) + \varepsilon(\vec{x})$

$$= f(\vec{a}) + \vec{\nabla} f(\vec{a}) \cdot (\vec{x} - \vec{a}) + \varepsilon(\vec{x})$$

with $\dfrac{|\varepsilon(\vec{x})|}{\|\vec{x} - \vec{a}\|} \to 0$ as $\vec{x} \to \vec{a}$.

Putting $\vec{x} = \vec{a} + t\vec{u}$, we have

$$f(\vec{a} + t\vec{u}) - f(\vec{a}) = \vec{\nabla} f(\vec{a}) \cdot t\vec{u} + \varepsilon(\vec{a} + t\vec{u})$$

$$\Rightarrow \quad \frac{f(\vec{a} + t\vec{u}) - f(\vec{a})}{t} = \vec{\nabla} f(\vec{a}) \cdot \vec{u} + \frac{\varepsilon(\vec{a} + t\vec{u})}{t}$$

Note that $|t| = \|\vec{x} - \vec{a}\|$,

$$\left| \frac{\varepsilon(\vec{a} + t\vec{u})}{t} \right| = \frac{|\varepsilon(\vec{a} + t\vec{u})|}{\|\vec{x} - \vec{a}\|} \to 0 \text{ as } t \to 0 \quad (\vec{x} = \vec{a} + t\vec{u})$$

$$\therefore \quad D_{\vec{u}} f(\vec{a}) = \lim_{t \to 0} \frac{f(\vec{a} + t\vec{u}) - f(\vec{a})}{t} = \vec{\nabla} f(\vec{a}) \cdot \vec{u}$$

#

Geometric Meanings of Gradient $\vec{\nabla} f$

> At a point $\vec{a}$, f <u>increases</u> (<u>decreases</u>) most <u>rapidly</u> in the <u>direction</u> of $\vec{\nabla} f(\vec{a})$ $(-\vec{\nabla} f(\vec{a}))$ at a <u>rate</u> of $\|\vec{\nabla} f(\vec{a})\|$

<u>Idea</u>: If f is differentiable at $\vec{a}$, then

$$D_{\vec{u}} f(\vec{a}) = \vec{\nabla} f(\vec{a}) \cdot \vec{u} \qquad (\text{for } \|\vec{u}\| = 1)$$

Cauchy-Schwarz $\Rightarrow$

$$|D_{\vec{u}} f(\vec{a})| \leq \|\vec{\nabla} f(\vec{a})\| \, \|\vec{u}\|$$

$$= \|\vec{\nabla} f(\vec{a})\|$$

i.e. $$-\|\vec{\nabla} f(\vec{a})\| \leq |D_{\vec{u}} f(\vec{a})| \leq \|\vec{\nabla} f(\vec{a})\|$$

"=" holds $\Leftrightarrow \vec{u} = -\dfrac{\vec{\nabla} f(\vec{a})}{\|\vec{\nabla} f(\vec{a})\|}$

"=" holds $\Leftrightarrow \vec{u} = \dfrac{\vec{\nabla} f(\vec{a})}{\|\vec{\nabla} f(\vec{a})\|}$

#

<u>Remark</u>: $D_{\vec{v}} f(\vec{a})$ can be defined for any vector $\vec{v}$, not necessary $\|\vec{v}\| = 1$ and could be $\vec{0}$, by the same definition

$$D_{\vec{v}} f(\vec{a}) = \lim_{t \to 0} \frac{f(\vec{a} + t\vec{v}) - f(\vec{a})}{t}$$

One can show that

$$D_{\vec{v}} f(\vec{a}) = \begin{cases} \|\vec{v}\| \, D_{\frac{\vec{v}}{\|\vec{v}\|}} f(\vec{a}), & \text{if } \vec{v} \neq \vec{0} \\ 0, & \text{if } \vec{v} = \vec{0} \end{cases}$$

and that

$$D_{\vec{v}} f(\vec{a}) = \vec{\nabla} f(\vec{a}) \cdot \vec{v} \qquad \text{if } f \text{ is differentiable at } \vec{a}$$

(not true in general, if f is not differentiable)
eg $f(x,y) = \sqrt{|xy|}$ at $(0,0)$

<u>Properties of Gradient</u>

If
- $f, g : \Omega \to \mathbb{R}$ ($\Omega \subset \mathbb{R}^n$, open) are differentiable,
- c is a constant,

then

(1) $\vec{\nabla}(f \pm g) = \vec{\nabla} f \pm \vec{\nabla} g$,

(2) $\vec{\nabla}(cf) = c\vec{\nabla} f$

(3) $\vec{\nabla}(fg) = g\vec{\nabla} f + f\vec{\nabla} g$

(4) $\vec{\nabla}\left(\frac{f}{g}\right) = \frac{g\vec{\nabla} f - f\vec{\nabla} g}{g^2}$ provided $g \neq 0$

(Pf: Easily from properties of partial derivatives)

Total Differential (of real-valued function)

$f: \Omega \to \mathbb{R}$ ($\Omega \subseteq \mathbb{R}^n$, open) differentiable at $\vec{a} \in \Omega$.

Then linearization at $\vec{a}$:

$$f(\vec{x}) = f(\vec{a}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})(x_i - a_i) + \varepsilon(\vec{x})$$

Usually denote: $\Delta f = f(\vec{x}) - f(\vec{a})$

$\Delta x_i = x_i - a_i$

Then $\Delta f \simeq \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a}) \Delta x_i$ $\left(\text{provided } \lim_{\vec{x}\to\vec{a}} \frac{|\varepsilon(\vec{x})|}{\|\vec{x}-\vec{a}\|} = 0\right)$

Classically, this approximation is presented as

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})\, dx_i \quad \left(\text{thinking: } \text{"}\Delta f \to df\text{"},\ \text{"}\Delta x_i \to dx_i\text{"}\right)$$

Def: Let
- $f: \Omega \to \mathbb{R}$, ($\Omega \subseteq \mathbb{R}^n$, open)
- $\vec{a} \in \Omega$

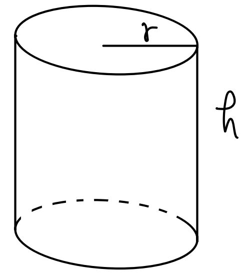
Suppose that f is differentiable on Ω. Then the total differential of f at $\vec{a}$ is defined to be the (formal) expression:

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})\, dx_i$$

Remark: In the future, df and dx_i can be interpreted as a linear maps from $\mathbb{R}^n$ to $\mathbb{R}$.

eg: Let $V(r, h) = \pi r^2 h$

(Volume of the Cylinder)



V is differentiable (because V is C^1)

$$dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh$$

$$= 2\pi r h\, dr + \pi r^2 dh$$

For application:

Suppose we want to approximate the change of V

when (r, h) changes $(r, h) = (3, 12)$ to $(3+0.08,\ 12-0.3)$

Then let $dr = \Delta r = 0.08$

$dh = \Delta h = -0.3$

we have

$$\Delta V \simeq dV = 2\pi r h\, dr + \pi r^2 dh$$

$$= 2\pi \cdot 3 \cdot 12 \cdot 0.08 + \pi 3^2 \cdot (-0.3)$$

$$= 3.06\pi \simeq 9.61$$

※

Properties of Total Differential

If $\begin{cases} \bullet\ f, g : \Omega \to \mathbb{R}\ (\Omega \subseteq \mathbb{R}^n \text{, open}) \text{ are differentiable, and} \\ \bullet\ c \in \mathbb{R} \text{ is a constant.} \end{cases}$

Then

(1) $d(f \pm g) = df \pm dg$,

(2) $d(cf) = c\,df$

(3) $d(fg) = g\,df + f\,dg$

(4) $d\left(\dfrac{f}{g}\right) = \dfrac{g\,df - f\,dg}{g^2}$ provided $g \neq 0$

(Pf: Easily from properties of partial derivatives)

Summary (on differentiation of a real-valued function on $\mathbb{R}^n$)

$f : \mathbb{R}^n \to \mathbb{R}$

A. Types of differentiations (derivatives)

- Directional Derivative :

$$D_{\vec{u}} f(\vec{a}) = \lim_{t \to 0} \frac{f(\vec{a} + t\vec{u}) - f(\vec{a})}{t} \qquad (\|\vec{u}\| = 1)$$

- Partial derivatives :

$$\frac{\partial f}{\partial x_i}(\vec{a}) = D_{\vec{e}_i} f(\vec{a}), \quad \vec{e}_i = (0, \cdots, \underset{\uparrow \text{ ith component}}{1}, \cdots 0)$$

- Gradient

$$\vec{\nabla} f(\vec{a}) = \left(\frac{\partial f}{\partial x_1}(\vec{a}), \cdots, \frac{\partial f}{\partial x_n}(\vec{a}) \right)$$

- Total Differential

$$df(\vec{a}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})\, dx_i$$

- Higher Derivatives

$$\frac{\partial^{k_1 + \cdots + k_n} f}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}(\vec{a})$$

(provided f is C^k, $k = k_1 + \cdots + k_n$)

$\uparrow$ all partial derivatives up to order k exist & cts.

B. Linear approximation

- $L(\vec{x}) = f(\vec{a}) + \vec{\nabla} f(\vec{a}) \cdot (\vec{x} - \vec{a})$

- $f(\vec{x}) = L(\vec{x}) + \varepsilon(\vec{x})$
  error term

- f is differentiable at $\vec{a}$

$$\Leftrightarrow \quad \lim_{\vec{x} \to \vec{a}} \frac{|\varepsilon(\vec{x})|}{\|\vec{x} - \vec{a}\|} = 0$$

In this case, $df \simeq \Delta f$ (by identifying $dx_i = \Delta x_i$)

C. Relations among various concepts

- $C^\infty \Rightarrow \cdots \Rightarrow C^{k+1} \Rightarrow C^k \Rightarrow \cdots \Rightarrow C^1 \Rightarrow C^0$ (No reverse implication)

- f is C^1 on an open set containing $\vec{a}$

$$\Downarrow \quad \not\Uparrow$$

f is differentiable at $\vec{a}$

$$\Swarrow \not\Nearrow \qquad\qquad \Searrow \not\Nwarrow$$

$D_{\vec{u}} f(\vec{a})$ exists $\forall \vec{u} \in \mathbb{R}^n$, $\|\vec{u}\| = 1$ $\quad \not\Rightarrow \;\; \not\Leftarrow \quad$ f is continuous at $\vec{a}$

$$\Searrow \not\Nwarrow \qquad\qquad \not\Swarrow \not\Nearrow$$

$\frac{\partial f}{\partial x_i}(\vec{a})$ exists, $\forall i = 1, \cdots, n$

Countex examples:

eg1: $f: \mathbb{R} \to \mathbb{R}$ (in MATH2050)

$$f(x)=\begin{cases} x^2 \sin\frac{1}{x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$$

f is differentiable on $\mathbb{R}$ but (check!)

$f'(x)$ is not continuous at $x=0$

(For $x\neq 0$, $f'(x)=2x\sin\frac{1}{x}-\cos\frac{1}{x}$ limit DNE as $x\to 0$)

Similarly $g(x)=x^{2k-2}f(x)$ is k-time differentiable but $g^{(k)}(x)$ is not continuous at $x=0$ (Pf: Omitted)

Hence k-time differentiable $\not\Rightarrow$ C^k.

(For multi-variable: $h(\vec{x})=h(x_1,\cdots,x_n)=g(x_1)$)

eg2

$$f(x,y)=\begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{if } x^2+y^2\neq 0 \\ 0 & \text{if } x^2+y^2=0 \end{cases}$$

$D_{\vec{u}}f(0,0)$ exists, $\forall$ unit vector $\vec{u}=(\cos\theta,\sin\theta)\in\mathbb{R}^2$

but f is not continuous at $(0,0)$ (check!)

eg3: $f(x,y)=|x+y|$ is continuous on $\mathbb{R}^2$, but

$f_x(0,0)$, $f_y(0,0)$ DNE (Check!)

eg 4 : $f(x,y) = \sqrt{|xy|}$

$f_x(0,0)$, $f_y(0,0)$ exist (in fact $=0$)

but $D_{\vec{u}} f(0,0)$ DNE for $\vec{u} \neq \pm\vec{e}_1, \pm\vec{e}_2$