A sufficient condition for differentiability:

> **Thm** Let $\Omega \subseteq \mathbb{R}^n$ be open, f be C^1 on Ω, then f is differentiable on Ω

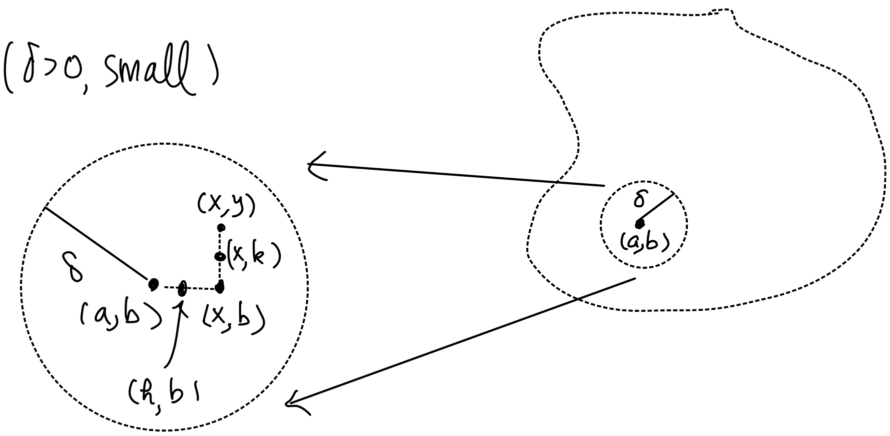
(The assumption requires all $\frac{\partial f}{\partial x_i}$ exist on Ω, not just at a single pt. $\vec{a}$)

Pf: (We prove it for 2-variables, similar proof for general case)

Suppose $(a,b) \in \Omega$

& $B_\delta(a,b) \subset \Omega$ ($\delta > 0$, small)

For any $(x,y) \in B_\delta(a,b)$



$$f(x,y) - f(a,b) = \underbrace{f(x,y) - f(x,b)} + \underbrace{f(x,b) - f(a,b)}$$

$$= f_y(x,k)(y-b) + f_x(h,b)(x-a) \quad \text{(by Mean Value Theorem)}$$

where k between y & b ; h between x & a.

$$\frac{|\varepsilon(x,y)|}{\|(x,y)-(a,b)\|} = \frac{|f(x,y) - f(a,b) - f_x(a,b)(x-a) - f_y(a,b)(y-b)|}{\|(x,y)-(a,b)\|}$$

$$= \frac{|f_y(x,k)(y-b) + f_x(h,b)(x-a) - f_x(a,b)(x-a) - f_y(a,b)(y-b)|}{\|(x,y)-(a,b)\|}$$

$$= \frac{\left| \left(f_x(h,b) - f_x(a,b)\right)(x-a) + \left(f_y(x,k) - f_y(a,b)\right)(y-b) \right|}{\|(x,y)-(a,b)\|}$$

$$\left(\begin{array}{c}\text{Cauchy-}\\ \text{Schwarz}\end{array}\right) \le \frac{\sqrt{\left(f_x(h,b)-f_x(a,b)\right)^2 + \left(f_y(x,k)-f_y(a,b)\right)^2}\ \sqrt{(x-a)^2+(y-b)^2}}{\|(x,y)-(a,b)\|}$$

$$= \sqrt{\left(f_x(h,b)-f_x(a,b)\right)^2 + \left(f_y(x,k)-f_y(a,b)\right)^2}$$

Note that if $(x,y) \to (a,b)$, then $(h,k) \to (a,b)$.

Hence

$$\frac{|\varepsilon(x,y)|}{\|(x,y)-(a,b)\|} \le \sqrt{\left(f_x(h,b)-f_x(a,b)\right)^2 + \left(f_y(x,k)-f_y(a,b)\right)^2} \to 0$$

as $(x,y) \to (a,b)$

because f_x & f_y are continuous.

$\therefore$ f is differentiable at (a,b).

Since $(a,b) \in \Omega$ is arbitrary, f is differentiable on Ω #

egs: (1) constant functions $f(\vec{x}) = c$ is differentiable

(2) coordinate functions $f(\vec{x}) = x_i$ are differentiable

(3) (1)&(2) $\Rightarrow$ $f(\vec{x}) = a + b_1x_1 + \cdots + b_nx_n$ is differentiable

(Linear function. For this f, what is the linearization $L(\vec{x})$ at $\vec{x} = \vec{0}$)

(4) Polynomials & rational functions are differentiable

(in their domain of definition).

(5) If $f(\vec{x})$ is differentiable, then $e^{f(\vec{x})}$, $\sin(f(\vec{x}))$, $\cos(f(\vec{x}))$

are differentiable.

And

$$\left.\begin{array}{ll} \ln(f(\vec{x})) & \text{when } f(\vec{x})>0 \\ \sqrt{f(\vec{x})} & \text{when } f(\vec{x})>0 \\ |f(\vec{x})| & \text{when } f(\vec{x})\neq 0 \\ \ln|f(\vec{x})| & \text{when } f(\vec{x})\neq 0 \end{array}\right\} \text{are differentiable.}$$

In particular, for example $\dfrac{e^{\sqrt{4+\sin(x^2+xy)}}}{\ln(1+\cos(x^2y))}$ is differentiable in the <u>domain of definition</u>

<u>eg</u>: $f(x,y,z) = xe^{x+y} - \ln(x+z) \quad \left(= xe^{x+y} - \log(x+z)\right)$

Domain of $f = \{(x,y,z)\in\mathbb{R}^3 : x+z>0\}$ (is open)

$$\frac{\partial f}{\partial x} = e^{x+y} + xe^{x+y} - \frac{1}{x+z}$$

$$\frac{\partial f}{\partial y} = xe^{x+y}$$

$x+z>0$ all terms are continuous in the domain of f.

$$\frac{\partial f}{\partial z} = -\frac{1}{x+z}$$

$\Rightarrow f$ is C^1 (on its domain) $\Rightarrow f$ is differentiable (on its domain).

Gradient and Directional Derivative

Def: Let
- $f : \Omega \to \mathbb{R}$, $(\Omega \subseteq \mathbb{R}^n$, open$)$
- $\vec{a} \in \Omega$

Then the <u>gradient vector</u> of f at $\vec{a}$ is defined to be

$$\vec{\nabla} f(\vec{a}) = \left(\frac{\partial f}{\partial x_1}(\vec{a}), \dots, \frac{\partial f}{\partial x_n}(\vec{a}) \right)$$

Remark: Using $\vec{\nabla} f$, linearization of f at $\vec{a}$ can be written as

$$\boxed{\begin{aligned} L(\vec{x}) &= f(\vec{a}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})(x_i - a_i) \\ &= f(\vec{a}) + \vec{\nabla} f(\vec{a}) \cdot (\vec{x} - \vec{a}) \end{aligned}}$$

eg: $f(x,y) = x^2 + 2xy$

$$\frac{\partial f}{\partial x} = 2x + 2y, \quad \frac{\partial f}{\partial y} = 2x$$

$$\therefore \vec{\nabla} f(x,y) = (2x+2y, 2x)$$

$$(\text{eg. } \vec{\nabla} f(1,2) = (6,2))$$

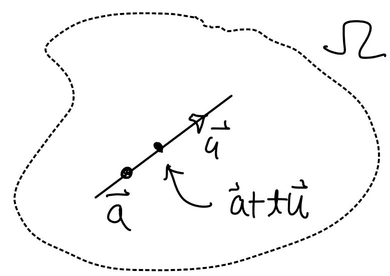
Def: Let
- $f: \Omega \to \mathbb{R}$, ($\Omega \subseteq \mathbb{R}^n$, open)
- $\vec{a} \in \Omega$
- $\vec{u} \in \mathbb{R}^n$ be a unit vector, i.e. $\|\vec{u}\| = 1$.

Then the directional derivative of f in the direction of $\vec{u}$ at $\vec{a}$ is defined to be

$$D_{\vec{u}} f(\vec{a}) = \lim_{t \to 0} \frac{f(\vec{a} + t\vec{u}) - f(\vec{a})}{t}$$

(= rate of change of f in the direction of $\vec{u}$ at the point $\vec{a}$)

Remark: If $\vec{u} = (0, \cdots, 1, \cdots, 0) = \vec{e}_j$,
j th component
$j = 1, \cdots, n$



$$D_{\vec{e}_j} f(\vec{a}) = \frac{\partial f}{\partial x_j}(\vec{a})$$