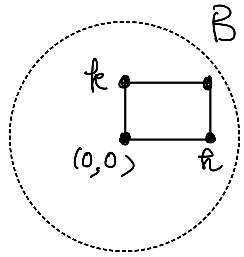
Pf of Clairaut's Thm



We may assume $\vec{a} = (0,0) \in \Omega$,

and we need to show

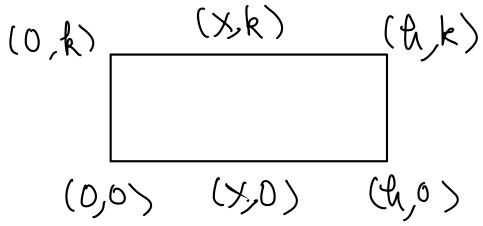
$$f_{xy}(0,0) = f_{yx}(0,0)$$

the open ball s.t. f_{xy}, f_{yx} both exist on B.

Let $h, k > 0$ and $[0,h] \times [0,k] \subset B \subset \Omega$

Define

$$\alpha = f(h,k) - f(h,0) - f(0,k) + f(0,0)$$



Let $g(x) = f(x,k) - f(x,0)$, $\quad 0 \le x \le h$

Then $\alpha = g(h) - g(0)$

$$g'(x) = f_x(x,k) - f_x(x,0)$$

Mean Value Thm $\Rightarrow \exists\, h_1 \in (0,h)$ s.t.

$$\frac{g(h) - g(0)}{h} = g'(h_1)$$

$$\Rightarrow \quad \frac{\alpha}{h} = f_x(h_1,k) - f_x(h_1,0)$$

i.e. $\quad \alpha = h\,[f_x(h_1,k) - f_x(h_1,0)]$

Mean Value Thm $\Rightarrow \exists\, k_1 \in (0,k)$ s.t.

$$\frac{f_x(h_1,k) - f_x(h_1,0)}{k} = (f_x)_y(h_1,k_1)$$

$\Rightarrow \quad \alpha = hk\, f_{xy}(h_1, k_1)$

Similarly, $\exists\ (h_2, k_2) \in (0,h)\times(0,k)$ s.t.

$$\alpha = hk\, f_{yx}(h_2, k_2) \qquad \text{(Ex!)}$$

$$\Rightarrow \quad f_{xy}(h_1,k_1) = f_{yx}(h_2,k_2)$$

Letting $h, k \to 0^+ \Rightarrow h_1, k_1 \to 0^+$, & $h_2, k_2 \to 0^+$

By continuity of f_{xy} & f_{yx} at $\vec{a}=(0,0)$, we have

$$f_{xy}(0,0) = f_{yx}(0,0)$$

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Def Let $f: \Omega \to \mathbb{R}$ ($\Omega \subseteq \mathbb{R}^n$, open)

Then • f is called a <u>C^k function</u> if <u>all partial derivatives</u> of f up to <u>order k</u> <u>exist</u> and are <u>continuous</u> on Ω

• f is called a <u>C^∞ function</u> if f is C^k for all $k \geq 0$.

<u>egs</u>: (1) If f is continuous (0-order partial derivative) then f is C^0.

(2) If f is C^2, then $f, f_x, f_y, f_{xx}, f_{xy} = f_{yx}, f_{yy}$ exist & are all continuous. (by Clairaut's)

(3) Polynomials, Rational functions, exponential, logarithm, trigonometric functions are C^∞ function on their domains of definition & hence their

sum / difference / product / quotient / compositions

are C^∞ function on their domains of definition.

explicit eg: $e^{x^2-y}\sin\left(\frac{x}{y}\right)$ is C^∞ on domain of definition $= \mathbb{R}^2 \setminus \{x\text{-axis}\}$
(except $y=0$)

<u>Generalization of Clairaut's Thm</u>

$\boxed{\text{If } f \text{ is } C^k \text{ on an open set } \Omega \subseteq \mathbb{R}^n\text{, then the order of (taking) differentiation \underline{does not matter} for all partial derivatives up to order } k.}$

<u>eg</u> If $f(x,y,z)$ is C^3, then

$$f_{xz} = f_{zx}\,, \quad f_{xyz} = f_{xzy} = f_{zxy} = f_{zyx}$$

$$\vdots \qquad\qquad = f_{yzx} = f_{yxz}$$

etc.

$$f_{xxy} = f_{xyx} = f_{yxx} \text{ and etc.}$$

Differentiability

Recall: 1-variable: f is differentiable at a

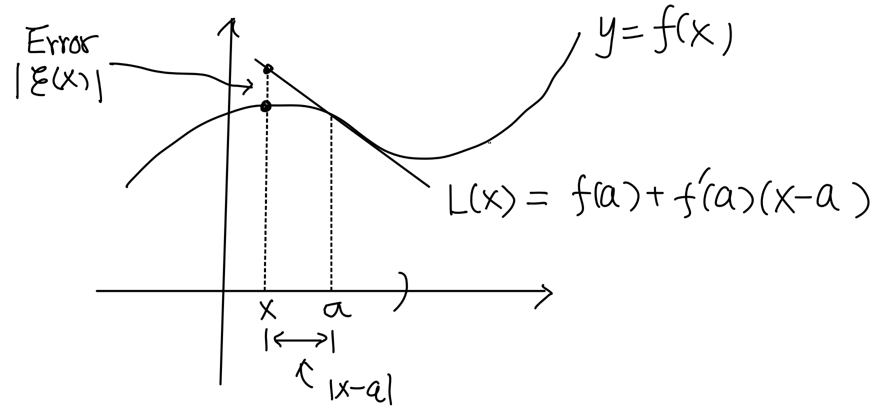
if $\quad f'(a) = \lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ exists

which is equivalent to

Linear Approximation of f at the point a:

$$f(x) \approx \underbrace{f(a) + f'(a)(x-a)}$$

$L(x)$ is the "best" linear function ($\deg \le 1$, poly) to approximate $f(x)$ near a



What does it mean by the "best"?

Answer: $$\lim_{x\to a} \frac{|f(x)-L(x)|}{|x-a|} = 0$$

where $f(x)-L(x)$ is usually referred as the

$$\boxed{\text{"error" term} \quad \mathcal{E}(x) = f(x) - L(x).}$$

$$\left(\lim_{x\to 0} \left| \frac{f(x)-f(a)}{x-a} - f'(a) \right| \;\; \Vert \;\; \lim_{x\to a} \frac{|\mathcal{E}(x)|}{|x-a|} = 0 \right)$$

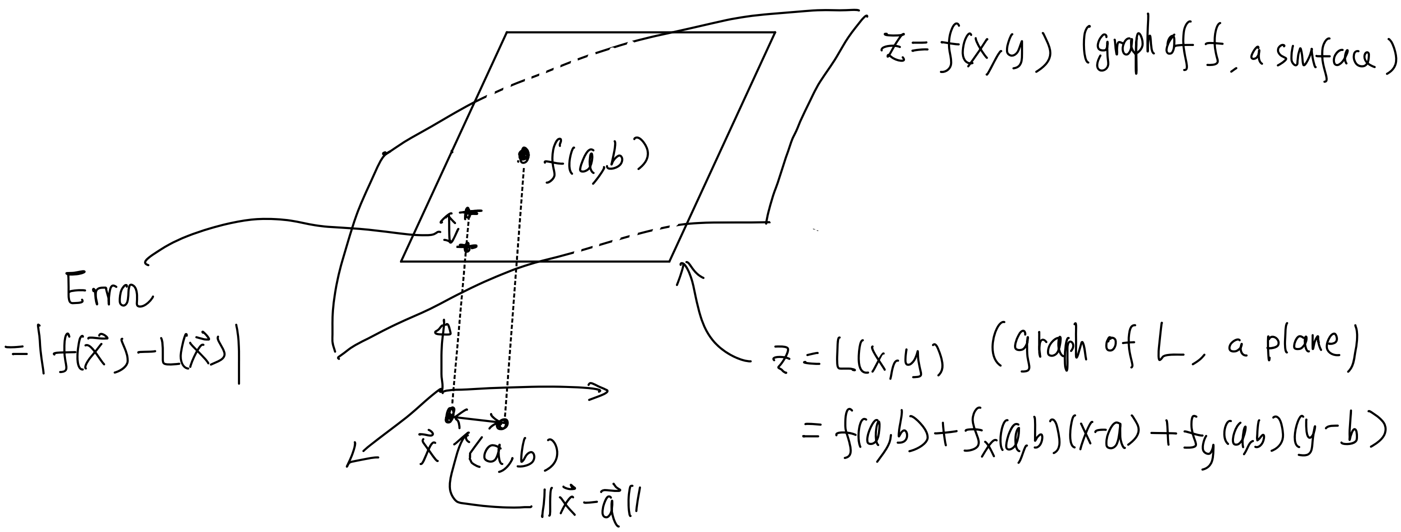
Higher dimension's analog:

linear function (deg ≤ 1, poly)

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

and want

$$f(x,y) \approx L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$



<u>Def</u>: Let
- $f : \Omega \to \mathbb{R}$, $\Omega \subseteq \mathbb{R}^n$, open
- $\vec{a} = (a_1, \cdots, a_n) \in \Omega$

Then f is said to be <u>differentiable at $\vec{a}$</u>

if (1) $\frac{\partial f}{\partial x_i}(\vec{a})$ exists for all $i = 1, \cdots, n$

(2) In the linear approximation for $f(\vec{x})$ at $\vec{a}$

$$f(\vec{x}) = \underbrace{f(\vec{a}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})(x_i - a_i)}_{L(\vec{x}) \text{ linear approx.}} + \underset{\substack{\uparrow \\ \text{error term}}}{\varepsilon(\vec{x})},$$

the <u>error term</u> $\varepsilon(\vec{x})$ satisfies

$$\lim_{\vec{x} \to \vec{a}} \frac{|\varepsilon(\vec{x})|}{\|\vec{x} - \vec{a}\|} = 0.$$

(A differentiable function is one which can be well approximated by a linear function locally.)

<u>Remark</u>: $L(\vec{x}) = f(\vec{a}) + \sum_{i=1}^{n} \underbrace{\frac{\partial f}{\partial x_i}(\vec{a})}_{\substack{\uparrow \\ \text{slope of } f \text{ in} \\ x_i\text{-direction at } \vec{a}}} \underbrace{(x_i - a_i)}_{\Delta x_i}$

- $L(\vec{x})$ is a deg ≤ 1 polynomial
- $L(\vec{a}) = f(\vec{a})$
- $\frac{\partial L}{\partial x_i}(\vec{a}) = \frac{\partial f}{\partial x_i}(\vec{a})$ (Easy Ex!)
- The graph of $y = L(\vec{x})$ is a n-plane <u>tangent</u> to the graph of $y = f(\vec{x})$ (which is a surface) at the point $\vec{x} = \vec{a}$.

<u>eg 1</u>: $f(x,y) = x^2 y$

(1) Show that f is differentiable at $(1,2)$

(2) Approximate $f(1.1, 1.9)$ using linearization.

(3) Find tangent plane of $z = f(x,y)$ at $(1, 2, \overset{f(1,2)}{2})$.

<u>Soln</u>: (1) $\frac{\partial f}{\partial x} = 2xy$, $\frac{\partial f}{\partial y} = x^2$

$\frac{\partial f}{\partial x}(1,2) = 4$, $\frac{\partial f}{\partial y}(1,2) = 1$

$\therefore$ The linearization at $(1,2)$ is

$$L(x,y) = f(1,2) + \frac{\partial f}{\partial x}(1,2)(x-1) + \frac{\partial f}{\partial y}(1,2)(y-2)$$

$$= 2 + 4(x-1) + (y-2) \quad (\text{or } = 4x+y-4)$$

With error term

$$\mathcal{E}(x,y) = f(x,y) - L(x,y)$$

$$= x^2y - [2 + 4(x-1) + (y-2)]$$

$$\lim_{(x,y)\to(1,2)} \frac{|\mathcal{E}(x,y)|}{\|(x,y)-(1,2)\|} = \lim_{(x,y)\to(1,2)} \frac{|x^2y - 2 - 4(x-1) - (y-2)|}{\sqrt{(x-1)^2+(y-2)^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{|(1+h)^2(k+2) - 2 - 4h - k|}{\sqrt{h^2+k^2}} \quad \left(\text{letting } \begin{array}{l} h = x-1 \\ k = y-2 \end{array}\right)$$

$$= \lim_{(h,k)\to(0,0)} \frac{|h^2k + 2hk + 2h^2|}{\sqrt{h^2+k^2}} \quad \left(\text{let: } \begin{array}{l} h = r\cos\theta \\ k = r\sin\theta \end{array}\right)$$

$$= \lim_{r\to 0} r\,|r\cos^2\theta\sin\theta + 2\cos\theta\sin\theta + 2\cos^2\theta|$$

$$= 0 \quad (\text{by Squeeze Thm})$$

$\therefore$ f is differentiable at $(1,2)$.

(b) $((1.1, 1.9) \approx (1,2))$

$$f(1.1, 1.9) \approx L(1.1, 1.9)$$

$$= 2 + 4(1.1-1) + (1.9-2)$$

$$= 2 + 4 \cdot 0.1 + (-0.1)$$

$$= 2.3$$

(c) The equation of the tangent plane of $z=f(x,y)$ at the point $(x,y)=(1,2)$ is

$$z = L(x,y) = 2+4(x-1)+(y-2)$$

$$\left(\text{i.e.} \quad z = 4x+y-4 \quad \text{or} \quad 4x+y-z=4 \right)$$

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eg 2 Is $f(x,y)=\sqrt{|xy|}$ differentiable at $(0,0)$?

Soln:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \cdots = 0 \qquad \text{(Similarly. Ex!)}$$

Linearization

$$L(x,y) = f(0,0) + \frac{\partial f}{\partial x}(0,0)\, x + \frac{\partial f}{\partial y}(0,0)\, y$$
$$= 0 + 0\cdot x + 0\cdot y$$
$$\equiv 0$$

Error term

$$\mathcal{E}(x,y) = f(x,y) - L(x,y) = f(x,y)$$
$$= \sqrt{|xy|}$$

$$\lim_{(x,y)\to(0,0)} \frac{|\mathcal{E}(x,y)|}{\|(x,y)-(0,0)\|} = \lim_{(x,y)\to(0,0)} \frac{\sqrt{|xy|}}{\sqrt{x^2+y^2}}$$

$$= \lim_{r\to 0} \frac{r\sqrt{|\cos\theta \sin\theta|}}{r} = \lim_{r\to 0} \sqrt{|\cos\theta\sin\theta|}$$

Different directions (i.e. different θ) give different limits.

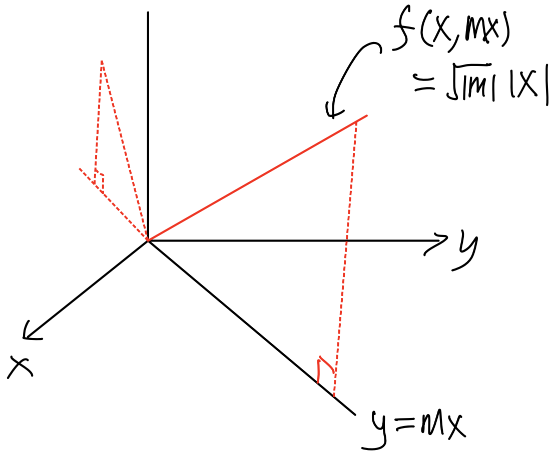
$$\therefore \lim_{(x,y)\to(0,0)} \frac{|\mathcal{E}(x,y)|}{\|(x,y)-(0,0)\|} \quad \text{DNE}.$$

$\therefore$ $f=\sqrt{|xy|}$ is <u>not</u> differentiable at $(0,0)$.

<u>Remark</u>: In this example, along the straight line $y=mx$

$$f(x,y)=\sqrt{|x\,mx|}=\sqrt{|m|}\;|x|$$

(only "approximated" by $L(x,y)$ in the $m=0$ situation.)



(<u>Note</u>: "Differentiability" $\Rightarrow$ we can approximate all (infinity many) directions by information along x & y direction. ($x_1,\cdots,x_n$) in general)

<u>Thm</u> If $f(\vec{x})$ is <u>differentiable</u> at $\vec{a}$, then $f(\vec{x})$ is <u>continuous</u> at $\vec{a}$.

<u>Pf</u>: $f(\vec{x})=L(\vec{x})+\varepsilon(\vec{x})$ is differentiable $\Leftrightarrow$ $\displaystyle\lim_{\vec{x}\to\vec{a}}\frac{|\varepsilon(\vec{x})|}{\|\vec{x}-\vec{a}\|}=0$

$$=f(\vec{a})+\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(\vec{a})(x_i-a_i)+\varepsilon(\vec{x})$$

$$\Rightarrow\ |f(\vec{x})-f(\vec{a})|\le\left|\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(\vec{a})(x_i-a_i)\right|+|\varepsilon(\vec{x})| \quad \text{(Triangle ineq.)}$$

$$\text{(Cauchy-Schwarz)}\ \le\left(\sqrt{\left(\frac{\partial f}{\partial x_i}(\vec{a})\right)^2}+\frac{|\varepsilon(\vec{x})|}{\|\vec{x}-\vec{a}\|}\right)\cdot\|\vec{x}-\vec{a}\|$$

$$\to 0 \quad \text{as } \vec{x}\to\vec{a}$$

by Squeeze Thm & Differentiability #

Thm If $f, g : \Omega \to \mathbb{R}$ ($\Omega \subseteq \mathbb{R}^n$, open) are differentiable at $\vec{a} \in \Omega$,

then (1) $f(\vec{x}) \pm g(\vec{x})$, $cf(\vec{x})$, $f(\vec{x})g(\vec{x})$ are differentiable at $\vec{a}$.

(2) $\dfrac{f(\vec{x})}{g(\vec{x})}$ is differentiable at $\vec{a}$ if $g(\vec{a}) \neq 0$

(3) (Special case of Chain Rule)

For 1-variable function $h(x)$ differentiable at $f(\vec{a})$, $h \circ f$ is differentiable at $\vec{a}$.