Higher Order Partial Derivatives

$n=2, \quad f(x,y) \longrightarrow \frac{\partial f}{\partial x}(x,y), \quad \frac{\partial f}{\partial y}(x,y)$ 1st order derivatives

$$\longrightarrow \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}(x,y)\right), \quad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x,y)\right)$$

2nd order derivatives

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right), \quad \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}f(x,y)\right)$$

be careful

Notations: $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = (f_x)_x = f_{xx}, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = (f_y)_x = f_{yx}$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = (f_x)_y = f_{xy}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = (f_y)_y = f_{yy}$$

Of course, similarly for 3rd order derivatives: eg

$$\frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\right]$$

$$= f_{yyx}$$

eg: $f(x,y) = x\sin y + y^2 e^{2x}$

Then
$$f_x = \sin y + 2y^2 e^{2x}$$
$$f_y = x\cos y + 2y e^{2x}$$
$$f_{xx} = (f_x)_x = (\sin y + 2y^2 e^{2x})_x = 4y^2 e^{2x}$$
$$f_{xy} = (f_x)_y = \cos y + 4y e^{2x}$$
$$f_{yx} = (f_y)_x = \cos y + 4y e^{2x}$$
$$f_{yy} = (f_y)_y = -x\sin y + 2e^{2x}$$

eg of higher order derivatives (order ≥ 3)

Previous eg : $f(x,y) = x\sin y + y^2 e^{2x}$

$\Rightarrow \quad f_{xy} = \cos y + 4y e^{2x} = f_{yx}$

$\Rightarrow$ (3rd order) $\quad f_{xyx} = (f_{xy})_x = 8ye^{2x} = (f_{yx})_x = f_{yxx}$

$f_{xyy} = (f_{xy})_y = -\sin y + 4e^{2x} = (f_{yx})_y = f_{yxy}$

$\vdots$

One can calculate similarly up to any order

Question Is it always true that $f_{xy} = f_{yx}$?

Answer : No.

Counterexample :

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

By definition

$$f_{xy}(0,0) = (f_x)_y(0,0) = \lim_{h\to 0} \frac{f_x(0,h) - f_x(0,0)}{h}$$

We need to calculate $f_x(0,h)$ ($h \neq 0$) & $f_x(0,0)$.

For $(0,h)$ $(h\neq 0)$

$$f_x(x,y) = \frac{\partial}{\partial x}\left(\frac{xy(x^2-y^2)}{x^2+y^2}\right) = \frac{(x^2+y^2)(3x^2y-y^3) - xy(x^2-y^2)(2x)}{(x^2+y^2)^2}$$

$$\Rightarrow \quad f_x(0,h) = \frac{-h^5}{h^4} = -h$$

For $(0,0)$, $\quad f_x(0,0) = \lim_{k\to 0} \frac{f(k,0)-f(0,0)}{k}$

$$= \lim_{k\to 0} \frac{0-0}{k} = 0$$

Hence $\quad f_{xy}(0,0) = \lim_{h\to 0} \frac{f_x(0,h)-f_x(0,0)}{h} = \lim_{h\to 0} \frac{-h-0}{h} = -1$

Similarly, $\quad f_y(h,0) = h \quad$ (Check!)

$\qquad f_y(0,0) = 0 \quad$ (Check!)

$$\Rightarrow \quad f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(h,0)-f_y(0,0)}{h} = \lim_{h\to 0} \frac{h-0}{h} = 1$$

(An easy way to see this is "$f(x,y) = -f(y,x)$")

Hence $\quad f_{xy}(0,0) \neq f_{yx}(0,0)$!

Question: When do we have $f_{xy} = f_{yx}$?

Thm (Clairaut's Thm / Mixed Derivatives Thm)

Let $f: \Omega \to \mathbb{R}$ ($\Omega \subset \mathbb{R}^n$, open)

If f_{xy} & f_{yx} exist and are continuous on Ω, then

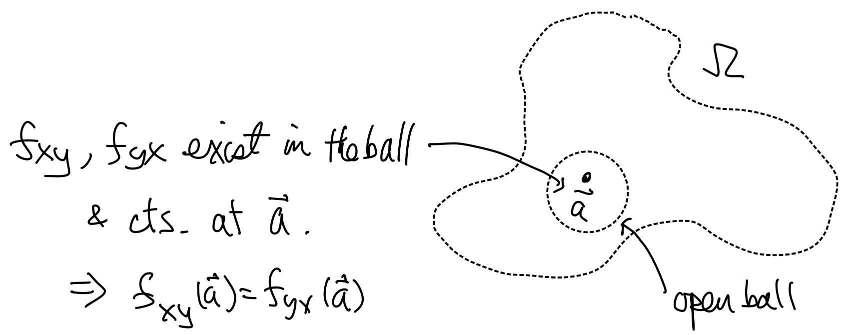
$$f_{xy} = f_{yx} \text{ on } \Omega.$$

Actually, one can prove a stronger version:

Thm Let $\begin{cases} \bullet\ f : \Omega \to \mathbb{R} \quad (\Omega \subset \mathbb{R}^n, \text{ open}) \\ \bullet\ \vec{a} \in \Omega \end{cases}$

If $\begin{cases} \bullet\ f_{xy}\ \&\ f_{yx} \text{ \underline{exist} in an open ball containing } \vec{a}, \text{ and} \\ \bullet\ f_{xy}\ \&\ f_{yx} \text{ are \underline{continuous} at } \vec{a}, \end{cases}$

then $f_{xy}(\vec{a}) = f_{yx}(\vec{a})$.



Recall **Mean Value Theorem** for 1-variable Function

Let $f : [a,b] \to \mathbb{R}$, $\begin{cases} \bullet \text{ continuous on } [a,b]\ \& \\ \bullet \text{ differentiable on } (a,b) \end{cases}$

Then $\exists\ c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$