eg: $\lim\limits_{(x,y)\to(0,0)} x\cos\left(\frac{1}{x^2+y^2}\right)$

Soln: $\left|x\cos\left(\frac{1}{x^2+y^2}\right)\right| = |x|\left|\cos\left(\frac{1}{x^2+y^2}\right)\right|$

$\leq |x|$

$\lim\limits_{(x,y)\to(0,0)} |x| = 0$, Squeeze Thm $\Rightarrow$ $\lim\limits_{(x,y)\to(0,0)} x\cos\left(\frac{1}{x^2+y^2}\right) = 0$

※

eg: $\lim\limits_{(x,y)\to(1,0)} \frac{(x-1)^2\ln x}{(x-1)^2+y^2}$ ($\ln x$ = natural log = $\log x$)

Soln: $\left|\frac{(x-1)^2\ln x}{(x-1)^2+y^2}\right| = \frac{(x-1)^2}{(x-1)^2+y^2}|\ln x| \leq |\ln x|$

$\lim\limits_{(x,y)\to(1,0)} |\ln x| = 0$ hence Squeeze Thm $\Rightarrow$

$\lim\limits_{(x,y)\to(1,0)} \frac{(x-1)^2\ln x}{(x-1)^2+y^2} = 0$. ※

Finding Limit using Polar Coordinates

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\Rightarrow$ " $(x,y)\to(0,0) \Leftrightarrow r\to 0$ "

eg Find limit $\lim\limits_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$ using polar coordinates.

Soln : Sub $x = r\cos\theta$, $y = r\sin\theta$, we have

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \lim_{r\to 0} \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2} = \lim_{r\to 0} r(\cos^3\theta + \sin^3\theta)$$

$$|r(\cos^3\theta + \sin^3\theta)| \le 2r \quad \& \quad \lim_{r\to 0} 2r = 0$$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = 0 \qquad \text{(by Squeeze Thm)}$$

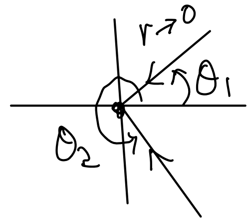
※

eg Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+xy}{2(x^2+y^2)}$ DNE.

Soln: $$\lim_{(x,y)\to(0,0)} \frac{x^2+xy}{2(x^2+y^2)} = \lim_{r\to 0} \frac{\cos^2\theta + \cos\theta\sin\theta}{2}$$

Different θ gives different limit $\Rightarrow \displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+xy}{2(x^2+y^2)}$ DNE.

(Different θ means
"approaching $(0,0)$ in different directions")



eg Find $\displaystyle\lim_{(x,y)\to(0,0)} xy\ln(x^2+y^2)$

Soln $$\lim_{(x,y)\to(0,0)} xy\ln(x^2+y^2) = \lim_{r\to 0} r^2\cos\theta\sin\theta\ln r^2$$

$$= \left(\lim_{r\to 0} 2r^2\ln r\right)\cos\theta\sin\theta = 0$$

(by L'Hopital's Rule)

(Also use Squeeze thm & $|2\cos\theta\sin\theta| \le 1$)

Iterated Limit

(1) $\lim_{x\to a}\lim_{y\to b} f(x,y) \overset{\text{def}}{=\!=} \lim_{x\to a}\left(\lim_{y\to b} f(x,y)\right)$

i.e. 1st take limit as $y\to b$, then take limit as $x\to a$.

(2) Similarly for $\lim_{y\to b}\lim_{x\to a} f(x,y)$

(3) Are they equal? $\left(\lim_{x\to a}\lim_{y\to b} f(x,y) = \lim_{y\to b}\lim_{x\to a} f(x,y)\ ?\right)$

(4) Relation to $\lim_{(x,y)\to(a,b)} f(x,y)$

eg: Consider $f(x,y) = \dfrac{x+y}{x-y}$

Soln

$$\lim_{x\to 0}\lim_{y\to 0}\frac{x+y}{x-y} = \lim_{x\to 0}\frac{x}{x} = 1$$

$$\lim_{y\to 0}\lim_{x\to 0}\frac{x+y}{x-y} = \lim_{y\to 0}\frac{y}{-y} = -1$$

both exist.

$\therefore \lim_{x\to a}\lim_{y\to b} f(x,y) \neq \lim_{y\to b}\lim_{x\to a} f(x,y)$

$\lim_{(x,y)\to(0,0)}\dfrac{x+y}{x-y}$ DNE (check! use "2 diff. paths provide 2 diff. limits")

Remarks

(eg $f(x,y)=\begin{cases} 1, & \text{if } x=y \\ 0, & \text{if } x\neq y \end{cases}$, $(a,b)=(0,0)$) (Ex!)

(1)

$$\lim_{x\to a}\lim_{y\to b} f(x,y) = \lim_{y\to b}\lim_{x\to a} f(x,y) \quad \not\Rightarrow \quad \lim_{(x,y)\to(a,b)} f(x,y)$$

both exist & equal $\quad \not\Leftarrow \quad$ exists (& equal)

(eg, $f(x,y)=\begin{cases} x\cos\frac{1}{y}+y\cos\frac{1}{x}, & \text{if } x,y\neq 0 \\ 0 & \text{if } x=0 \text{ or } y=0 \end{cases}$)

$(a,b)=(0,0)$

(Optional Ex!)

(2) If all 3 limits exist, then they are equal!

Continuity

Def: Let $f : A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ & $\vec{a} \in A$ (so $f(\vec{a})$ is defined)

Then f is said to be continuous at $\vec{a}$

if $\boxed{\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = f(\vec{a})}$ (exists & equal to $f(\vec{a})$.)

Equivalently, $\forall \varepsilon > 0, \exists \delta > 0$ such that

if $\vec{x} \in A$ & $\|\vec{x} - \vec{a}\| < \delta$, then $|f(\vec{x}) - f(\vec{a})| < \varepsilon$.

Def $f : A \to \mathbb{R}$ is said to be continuous (on A) if

f is continuous at every point in A.