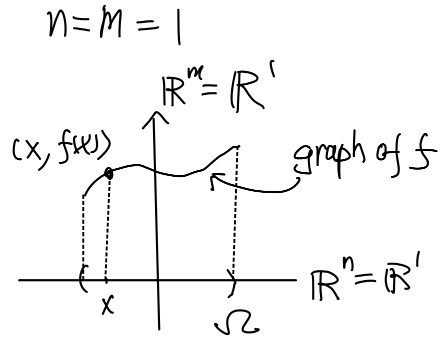
Vector-valued functions of Multivariables

$$\vec{f} : \Omega^{\subset \mathbb{R}^n} \longrightarrow \mathbb{R}^m \qquad \text{How to visualize it ?}$$

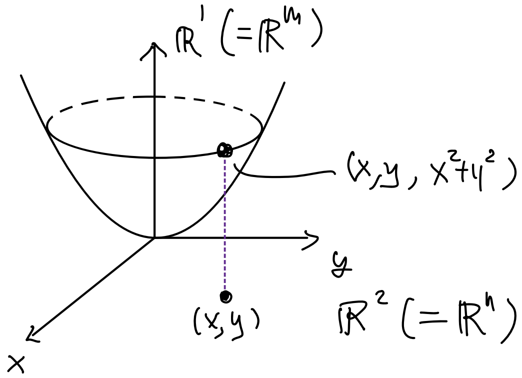
(1) Graph of $\vec{f}$



$$\mathrm{Graph}(\vec{f}) = \{ (\underset{\mathbb{R}^n}{\underset{\cap}{\vec{x}}}, \underset{\mathbb{R}^m}{\underset{\cap}{\vec{f}(\vec{x})}}) : \vec{x} \in \Omega \}$$

$$\subseteq \mathbb{R}^{n+m}$$

<u>eg</u> : $n = 2$, $m = 1$: $g : \mathbb{R}^2 \to \mathbb{R}$ defined $g(x,y) = x^2 + y^2$



graph (g) is the surface

$$= \{ (x, y, x^2+y^2) \in \mathbb{R}^3 : (x,y) \in \mathbb{R}^2 \} \subseteq \mathbb{R}^3$$

(In general, it is impossible to draw the graph for $n + m > 3$!)

(2) Level set of $\vec{f} : \Omega^{\subset \mathbb{R}^n} \longrightarrow \mathbb{R}^m$

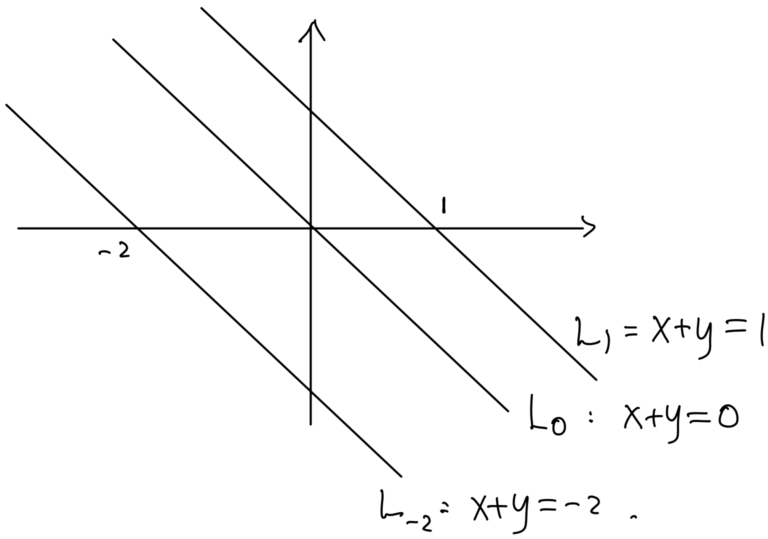
If $\vec{c} \in \mathbb{R}^m$, define the level set at $\vec{c}$ to be

$$L_{\vec{c}} = \{ \vec{x} \in \Omega \subset \mathbb{R}^n : \vec{f}(\vec{x}) = \vec{c} \} = (\vec{f})^{-1}(\vec{c}) \subseteq \Omega \subseteq \mathbb{R}^n.$$

eg: $f(x,y) = x+y$, $\Omega = \mathbb{R}^2$ ($m=1 \Rightarrow \vec{c} \in \mathbb{R}^1$, ie $\vec{c}$ is a number)

$L_c = \{(x,y) \in \mathbb{R}^2 : f(x,y) = c\}$

$= \{(x,y) \in \mathbb{R}^2 : x+y = c\}$



eg: $g(x,y) = x^2+y^2$, $\Omega = \mathbb{R}^2$

$L_c = \{(x,y) \in \mathbb{R}^2 : g(x,y) = c\}$

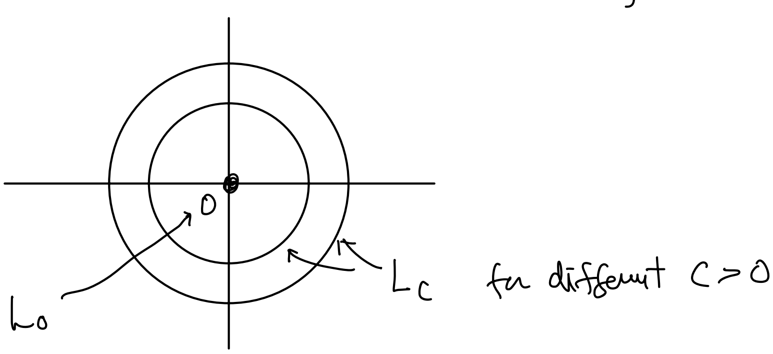
$= \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = c\}$

Case 1: $c < 0$, $L_c = \emptyset$

Case 2: $c = 0$, $L_0 = \{(0,0)\}$

Case 3: $c > 0$, $L_c = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = c\}$

= circle of radius $\sqrt{c}$ centered at $(0,0)$.



eg: $h(x,y) = \cos(2\pi(x^2+y^2))$, $\Omega = \mathbb{R}^2$

$$L_c = \{(x,y) \in \mathbb{R}^2 : h(x,y) = c\}$$
$$= \{(x,y) \in \mathbb{R}^2 : \cos(2\pi(x^2+y^2)) = c\}$$

Case 1: If $|c| > 1$, then $L_c = \phi$

Case 2: If $|c| \leq 1$, then $L_c = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = \frac{1}{2\pi}\cos^{-1}(c)\}$

subcase (a) $\cos^{-1}(c) < 0$, $L_c = \phi$

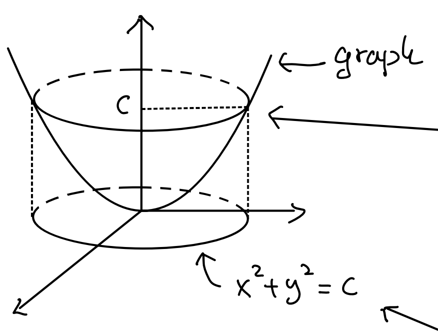
subcase (b) $\cos^{-1}(c) = 0$, $L_c = \{(0,0)\}$

subcase (c) $\cos^{-1}(c) > 0$, L_c = circle of radius $\sqrt{\frac{1}{2\pi}\cos^{-1}(c)}$ centered at $(0,0)$.

(depends on how you choose $\cos^{-1}(c)$, we may choose $\cos^{-1}$ s.t. subcase (a) will not occur. Further discussion omitted)

Level set $\leftrightarrow$ graph

$g(x,y) = x^2+y^2$



cut at horizontal level

$z = c \ (\geq 0)$

$\downarrow$ project down

level set.

Limit of Multi-variable Functions

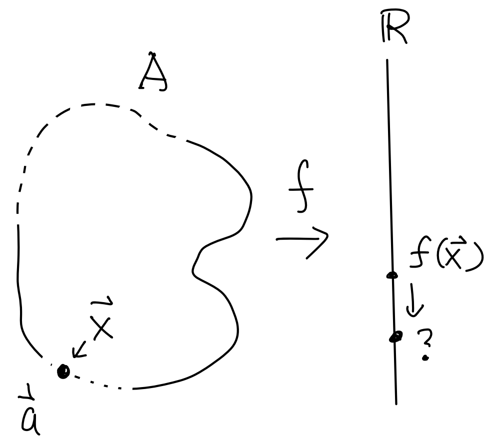
Let $A \subseteq \mathbb{R}^n$

$f : A \to \mathbb{R}$ be a function (of n-variables)

Let $\overline{A} \overset{\text{def}}{=\!=} A \cup \partial A$ the <u>closure</u> of A

For $\vec{a} \in A \cup \partial A$, we consider

$$\lim_{\vec{x} \to \vec{a}} f(\vec{x})$$



In general n, m-dims

> **Def** (ε-δ) : Let $\vec{f} : A \subset \mathbb{R}^n \to \mathbb{R}^m$, $\vec{a} \in \overline{A} = A \cup \partial A$
>
> We say that $\displaystyle \lim_{\vec{x} \to \vec{a}} \vec{f}(\vec{x}) = \vec{L}$
>
> if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that
>
> $\vec{x} \in A$ and $0 < \|\vec{x} - \vec{a}\| < \delta \Rightarrow \|\vec{f}(\vec{x}) - \vec{L}\| < \varepsilon$

<u>Remarks</u> (i) $\|\vec{x} - \vec{a}\|$ = distance between $\vec{x}$ and $\vec{a}$ in $\mathbb{R}^n$

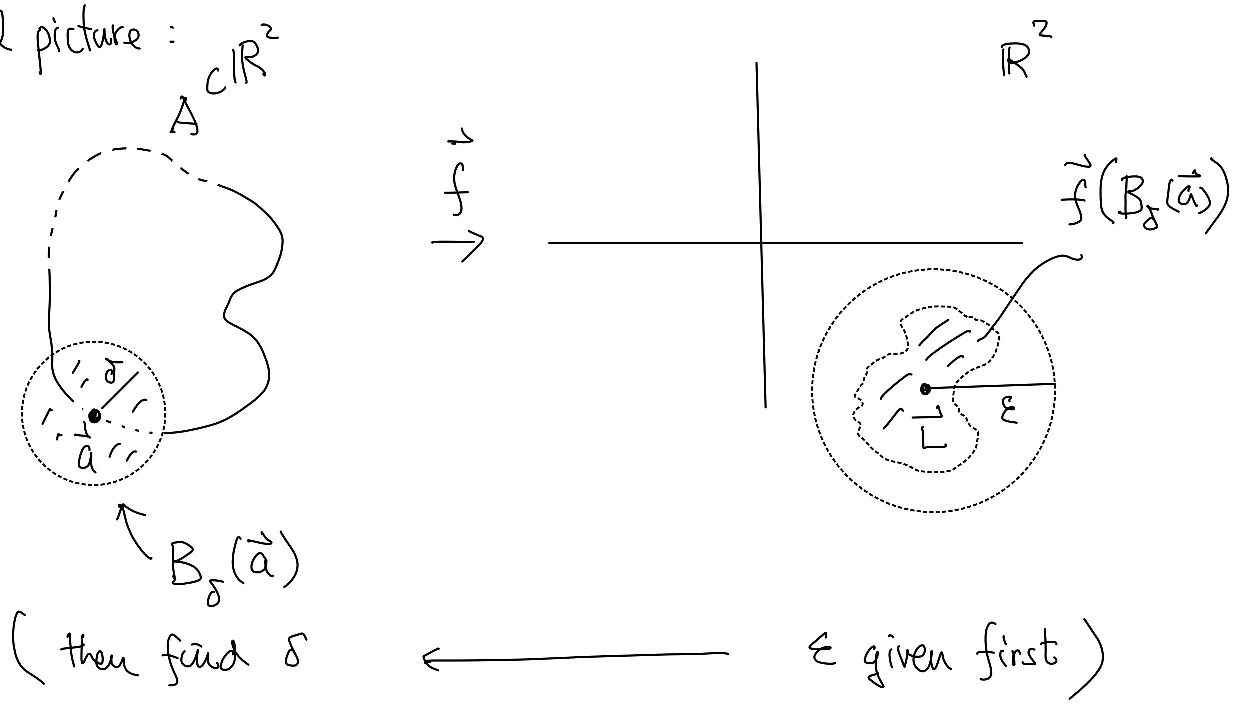
$0 < \|\vec{x} - \vec{a}\|$ means $\vec{x} \neq \vec{a}$

i.e. Considering points close to $\vec{a}$ but <u>not equal</u> to $\vec{a}$.

(ii) $\|\vec{f}(\vec{x}) - \vec{L}\|$ = distance between $\vec{f}(\vec{x})$ and $\vec{L}$ in $\mathbb{R}^m$.

If $m = 1$, $\|\vec{f}(\vec{x}) - \vec{L}\| = |f(\vec{x}) - L|$ absolute value of the different.

2 dim'l picture :



eg : $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x+y$

Illustrate that $\displaystyle\lim_{(x,y)\to(1,2)} f(x,y) = 3$.

Soln : [i.e. you need to show that given any $\varepsilon > 0$, one can find $\delta > 0$ such that if $0 < \|(x,y)-(1,2)\| < \delta$ then $|f(x,y)-3| < \varepsilon$. (No need to check $(x,y) \in A$, because $A = \mathbb{R}^2$)]

Idea : $|f(x,y)-3| = |x+y-3|$

$= |(x-1)+(y-2)| \leq |x-1| + |y-2|$

$\|(x,y)-(1,2)\| = \sqrt{(x-1)^2+(y-2)^2}$

For instance, for $\varepsilon = 1$, choose $\delta = \frac{1}{2}$

if $\|(x,y)-(1,2)\| < \delta = \frac{1}{2}$, then

$|x-1| \leq \sqrt{(x-1)^2+(y-2)^2} < \frac{1}{2}$

$|y-2| \leq \sqrt{(x-1)^2+(y-2)^2} < \frac{1}{2}$

& hence $|f(x,y)-3| \leq |x-1| + |y-2| < \frac{1}{2} + \frac{1}{2} = 1 = \varepsilon$.

Similarly, for $\varepsilon = \frac{1}{100}$, one can choose $\delta = \frac{1}{200}$. (Ex!)

<u>(Real) Proof:</u> For any given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{2}$. Then

$$\|(x,y)-(1,2)\| < \delta = \frac{\varepsilon}{2}$$

$$\Rightarrow |f(x,y)-3| = |x+y-3| = |(x-1)+(y-2)| \leq |x-1| + |y-2|$$
$$< \delta + \delta = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(Since $|x-1| \leq \|(x,y)-(1,2)\|$ & $|y-2| \leq \|(x,y)-(1,2)\|$)

$$\therefore \lim_{(x,y)\to(1,2)} f(x,y) = 3$$ ※

<u>eg</u>: Let $f(x,y) = x^2+y^2$

Show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ from definition.

<u>Soln</u>: Need to show that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

if $0 < \|(x,y)-(0,0)\| = \sqrt{x^2+y^2} < \delta$

then $\|f(x,y)-0\| = |x^2+y^2| < \varepsilon$

<u>eg</u>: $\varepsilon = \frac{1}{100}$, then $\delta = \sqrt{\varepsilon} = \frac{1}{10}$.

If $\|(x,y)-(0,0)\| < \delta = \frac{1}{10}$, then $\sqrt{x^2+y^2} < \frac{1}{10}$

$\Rightarrow$ $x^2+y^2 < \frac{1}{100}$, i.e. $\|f(x,y)-0\| < \frac{1}{100} = \varepsilon$

(Real) Proof: $\forall \varepsilon > 0$, choose $\delta = \sqrt{\varepsilon} > 0$.

$$0 < \|(x,y)-(0,0)\| < \delta = \sqrt{\varepsilon} \Rightarrow \sqrt{x^2+y^2} < \sqrt{\varepsilon}$$

$$\Rightarrow x^2+y^2 < \varepsilon \text{ , i.e. } \|f(x,y)-0\| = x^2+y^2 < \varepsilon .$$

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = 0 .$$

Prop: Let
- $A \subseteq \mathbb{R}^n$
- $\vec{a} \in \overline{A} = A \cup \partial A$
- $\vec{f} : A \to \mathbb{R}^m$ with

$$\vec{f}(\vec{x}) = \begin{bmatrix} f_1(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{bmatrix} = \begin{bmatrix} f_1(x_1,\cdots,x_n) \\ \vdots \\ f_m(x_1,\cdots,x_n) \end{bmatrix}$$

where $\vec{x} = (x_1, \cdots, x_n) \in A$.

Then

$$\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) = \vec{L} = \begin{bmatrix} l_1 \\ \vdots \\ l_m \end{bmatrix} \Leftrightarrow \lim_{\vec{x}\to\vec{a}} f_i(\vec{x}) = l_i , \ \forall i = 1,\cdots,m.$$

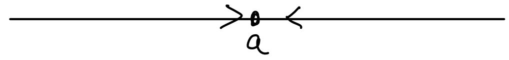
Consequence: It is good enough for us to focus on limit of real-valued function $f: A \to \mathbb{R}$ (i.e. $m=1$)

eg: $\vec{f}: \mathbb{R}^2 \to \mathbb{R}^2$ $\vec{f}(x,y) = \begin{bmatrix} x+y \\ x^2+y^2+1 \end{bmatrix}$

$$\lim_{(x,y)\to(1,2)} \vec{f}(x,y) = \begin{bmatrix} \lim_{(x,y)\to(1,2)} (x+y) \\ \lim_{(x,y)\to(1,2)} (x^2+y^2+1) \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \quad (\leftarrow \text{Ex!})$$

Limit along a path

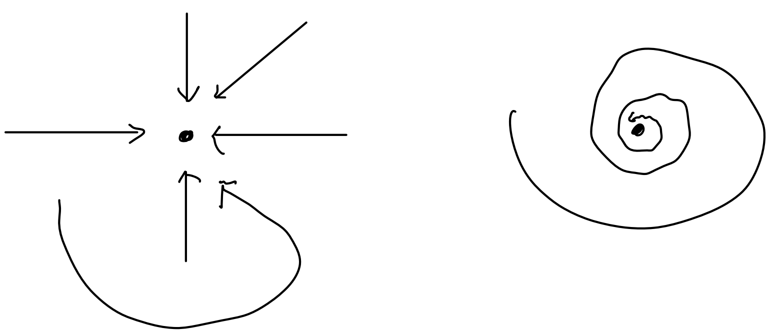
Recall: In one variable:



$$\lim_{x\to a} f(x) \text{ exists} \iff \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$

(exist & equal)

For n-variables, $n \geq 2$, there are infinitely many ways to approach a point in $\mathbb{R}^n$. Situation is very complicated.



However, we still have the following

Fact: $f: A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$, $\vec{a} \in \bar{A} = A \cup \partial A$. Then

$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L \iff$ limit of $f(\vec{x})$ when $\vec{x}$ approaches to $\vec{a}$ along any curve (path) exists and equals to L

- Useful for showing limit "does not exists" (DNE) (only in our dept. not a common notation)

(i) Find a path such that the limit along that path DNE, or

(ii) Find 2 paths such that the limits along the 2 paths are different

$\Rightarrow \lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x})$ DNE.

eg $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ $\left(\frac{x^2-y^2}{x^2+y^2} \text{ doesn't define at } (x,y)=(0,0) \right)$

Soln: (1) Along x-axis $(y=0)$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

(2) Along y-axis $(x=0)$

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{x^2-y^2}{x^2+y^2} = \lim_{y\to 0} \frac{-y^2}{y^2} = -1$$

Different limits along different paths $\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ DNE.

(In fact, we can try other paths too, for instance, $y=x$

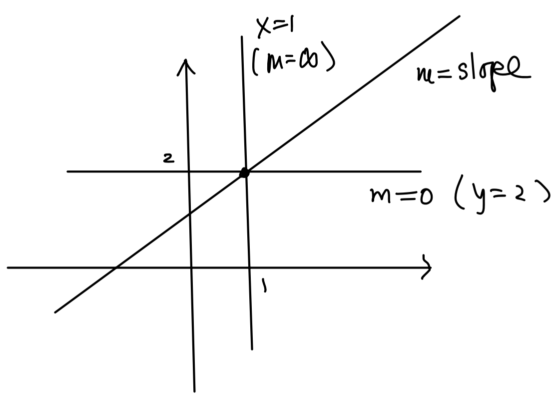
$$\lim_{\substack{(x,y)\to(0,0)\\ x=y}} \frac{x^2-y^2}{x^2+y^2} = 0 \quad (\text{Ex!})$$
)

eg: Consider $\lim_{(x,y)\to(1,2)} \frac{xy-2x-y+2}{(x-1)^2+(y-2)^2}$ along all straight lines passing through $(1,2)$.

Soln: (1) Along $x=1$

$$\lim_{\substack{(x,y)\to(1,2)\\ x=1}} \frac{xy-2x-y+2}{(x-1)^2+(y-2)^2}$$

$$= \lim_{y\to 2} \frac{y-2-y+2}{(y-2)^2} = 0$$



(2) Along the line with slope $= m$ & passing thro. $(1,2)$

$$y-2 = m(x-1)$$

$$\lim_{\substack{(x,y)\to(1,2)\\ y-2=m(x-1)}} \frac{xy-2x-y+2}{(x-1)^2+(y-2)^2} = \lim_{\substack{(x,y)\to(1,2)\\ y-2=m(x-1)}} \frac{(x-1)(y-2)}{(x-1)^2+(y-2)^2} \quad (\text{check!})$$

$$= \lim_{x\to 1} \frac{(x-1)\cdot m(x-1)}{(x-1)^2+(m(x-1))^2} = \lim_{x\to 1} \frac{m}{1+m^2}$$

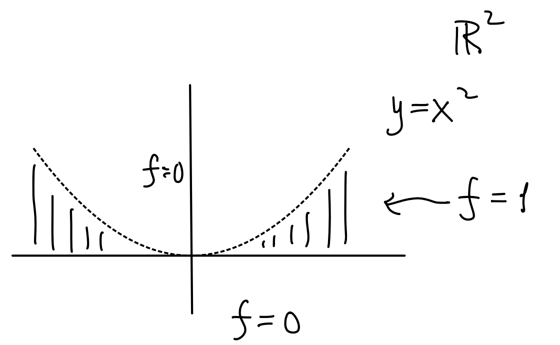
$$= \frac{m}{1+m^2}.$$

(Different limits for different slopes (ie different paths)

$\therefore \displaystyle\lim_{(x,y)\to(1,2)} \frac{xy-2x-y+2}{(x-1)^2+(y-2)^2}$ DNE.)

eg: $f: \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} 1, & \text{if } 0<y<x^2 \\ 0, & \text{otherwise} \end{cases}$$



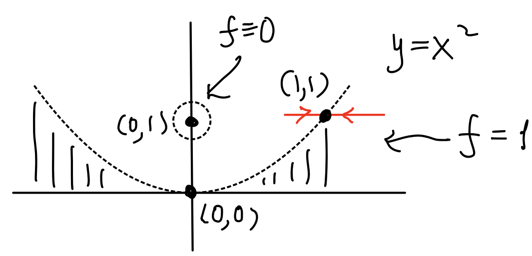
($f \equiv 1$ for |||....|||, 0 other places)

Find $\displaystyle\lim_{(x,y)\to\vec{a}} f(x,y)$, where

(i) $\vec{a} = (0,1)$

(ii) $\vec{a} = (1,1)$

(iii) $\vec{a} = (0,0)$



Soln: (i) For $\vec{a}=(0,1)$, $f(x,y)=0$ near $(0,1)$ $\Rightarrow \lim\limits_{(x,y)\to(0,1)} f(x,y)=0$.

(ii) For $\vec{a}=(1,1)$,

$$\lim_{\substack{(x,y)\to(1,1)\\ x<1,\, y=1}} f(x,y) = \lim_{\substack{x\to 1^-\\ (y=1)}} 0 = 0$$

$$\neq$$

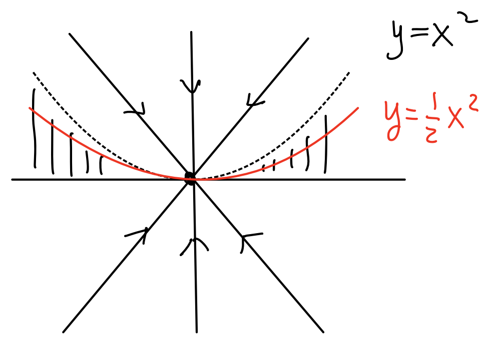
$$\lim_{\substack{(x,y)\to(1,1)\\ x>1,\, y=1}} f(x,y) = \lim_{\substack{x\to 1^+\\ (y=1)}} 1 = 0\ 1$$

Different limits for different paths $\Rightarrow \lim\limits_{(x,y)\to(1,1)} f(x,y)$ DNE

(iii) Case 1 Along y-axis $(x=0)$

$f(0,y)=0$, $\forall y$

$$\Rightarrow \lim_{\substack{(x,y)\to(0,0)\\ x=0}} f(x,y)=0$$



Case 2: Along $y=mx$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} f(x,y) = \lim_{x\to 0} f(x,mx)$$

$$= 0.$$

Case 3 Along the curve $y=\frac{1}{2}x^2$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=\frac{1}{2}x^2}} f(x,y) = \lim_{x\to 0} f(x,\tfrac{1}{2}x^2) = \lim_{x\to 0} 1 = 1$$

$\therefore$ Case 3 & Case 2 together $\Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y)$ DNE. ※

Properties of Limits

Assuming all limits on the right hand side exist, then the limit on the left hand side exists and the formula holds

(1) $$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) \pm g(\vec{x}) = \lim_{\vec{x}\to\vec{a}} f(\vec{x}) \pm \lim_{\vec{x}\to\vec{a}} g(\vec{x})$$

(2) $$\lim_{\vec{x}\to\vec{a}} k f(\vec{x}) = k \lim_{\vec{x}\to\vec{a}} f(\vec{x})$$, where k is a constant

(3) $$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) g(\vec{x}) = \lim_{\vec{x}\to\vec{a}} f(\vec{x}) \lim_{\vec{x}\to\vec{a}} g(\vec{x})$$

(4) $$\lim_{\vec{x}\to\vec{a}} \frac{f(\vec{x})}{g(\vec{x})} = \frac{\lim_{\vec{x}\to\vec{a}} f(\vec{x})}{\lim_{\vec{x}\to\vec{a}} g(\vec{x})}$$ if $\lim_{\vec{x}\to\vec{a}} g(\vec{x}) \neq 0$

(5) $$\lim_{\vec{x}\to\vec{a}} \left[f(\vec{x})\right]^n = \left[\lim_{\vec{x}\to\vec{a}} f(\vec{x})\right]^n$$, $n \geq 0$ (integer)

(6) $$\lim_{\vec{x}\to\vec{a}} \left[f(\vec{x})\right]^{\frac{1}{n}} = \left[\lim_{\vec{x}\to\vec{a}} f(\vec{x})\right]^{\frac{1}{n}}$$, $n \geq 0$ (integer)

(If n is even, assume $f(\vec{x}) \geq$ near $\vec{a}$.)

Squeeze Theorem (Sandwich Theorem)

Let $f, g, h : \Omega \overset{\subset \mathbb{R}^n}{\longrightarrow} \mathbb{R}$ be functions of n-variables

If $\begin{cases} \bullet\ g(\vec{x}) \le f(\vec{x}) \le h(\vec{x}) \text{ near } \vec{a} \in \Omega \text{ and} \\ \bullet\ \lim_{\vec{x} \to \vec{a}} g(\vec{x}) = \lim_{\vec{x} \to \vec{a}} h(\vec{x}) = L. \end{cases}$

Then $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L$

Special Case of Squeeze Theorem

If $\begin{cases} \bullet\ |f(\vec{x})| \le g(\vec{x}) \text{ near } \vec{a} \text{ and} \\ \bullet\ \lim_{\vec{x} \to \vec{a}} g(\vec{x}) = 0 \end{cases}$

Then $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = 0$.

$\left(|f| \le g \Rightarrow -g \le f \le g \right)$