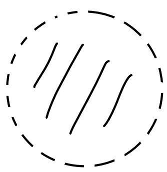
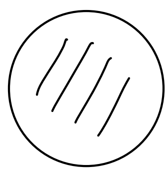
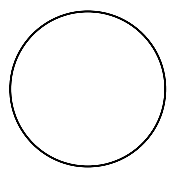
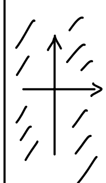
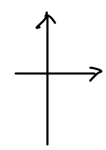
Eg:

Subset $S \subset \mathbb{R}^2$	$B_1(0,0) = \{(x,y) \in \mathbb{R}^2 : x^2+y^2<1\}$	$\overline{B_1(0,0)} = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$	$S^1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$	$\mathbb{R}^2$	ϕ (empty set)
$\text{Int}(S)$	$B_1(0,0)$	$B_1(0,0)$	ϕ	$\mathbb{R}^2$	ϕ
$\text{Ext}(S)$	$\mathbb{R}^2 \setminus \overline{B_1(0,0)} = \{(x,y) \in \mathbb{R}^2 : x^2+y^2>1\}$	$\mathbb{R}^2 \setminus \overline{B_1(0,0)}$	$\mathbb{R}^2 \setminus S^1$	ϕ	$\mathbb{R}^2$
∂S	S^1	S^1	S^1	ϕ	ϕ
Open?	Yes	No	No	Yes	Yes
Closed?	No	Yes	Yes	Yes	Yes
Picture					

Remarks: (1) There are exactly two subsets of $\mathbb{R}^n$ which are both open and closed: $\mathbb{R}^n$ and ϕ.

(2) Some subsets of $\mathbb{R}^n$ are neither open nor closed. (eg: above)

(3) For any $S \subseteq \mathbb{R}^n$, $\text{Int}(S)$ & $\text{Ext}(S)$ are open in $\mathbb{R}^n$;
∂S is closed in $\mathbb{R}^n$

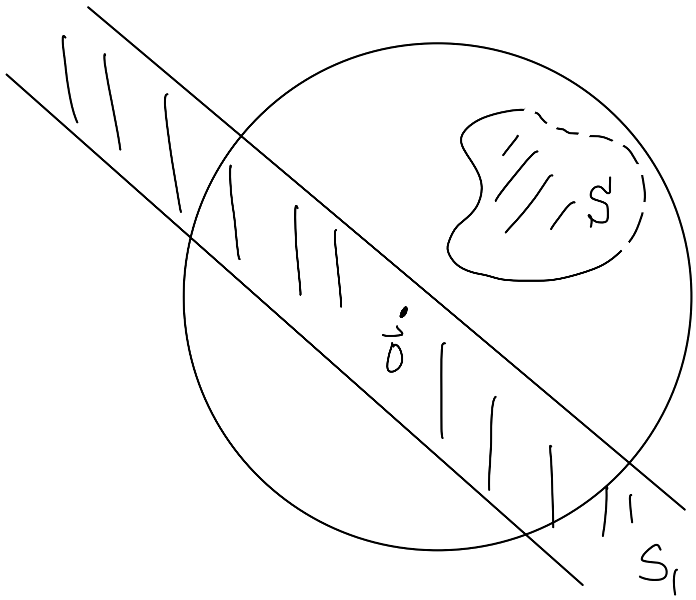
(Ex: What about $\text{Int}(S) \cup \partial S$?)

Def: $S \subseteq \mathbb{R}^n$ is called <u>bounded</u> if

$\exists M > 0$ such that

$$S \subseteq B_M(\vec{0}) = \{ \vec{x} \in \mathbb{R}^n : \|\vec{x}\| < M \}$$

S is called <u>unbounded</u> if it is not bounded



S is <u>bounded</u>

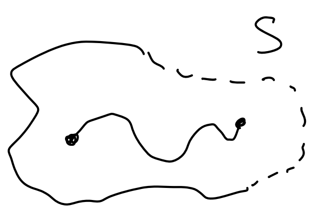
S_1 is <u>unbounded</u>

<u>eg</u>: $y\text{-axis} = \{ (x,y) \in \mathbb{R}^2 : x = 0 \}$ is unbounded

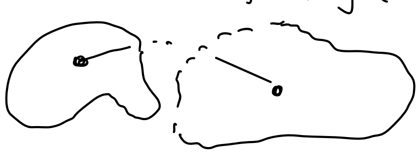
(Pf: $\forall M > 0$, $\exists$ $(0, 2M) \in y\text{-axis}$ s.t. $(0, 2M) \notin B_M(\vec{0})$.)

Def $S \subset \mathbb{R}^n$ is called <u>path-connected</u> if any two points in S can be connected by a curve in S.

path-connected

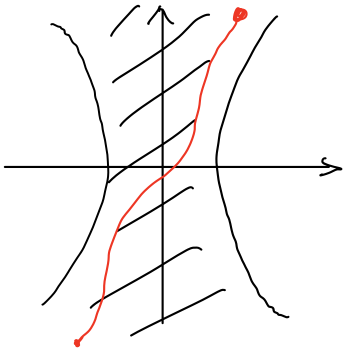


cannot joint by a curve completely inside S.

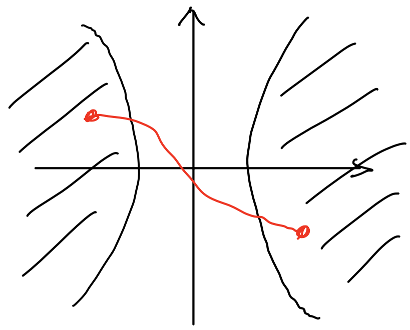


eg: $S = \{(x,y)\in\mathbb{R}^2 : x^2-y^2 \le 1\}$ is path-connected

$S_1 = \{(x,y)\in\mathbb{R}^2 : x^2-y^2 \ge 1\}$ is not path-connected.



S



S_1

Remark: In topology, there is a different notion called "connected". We'll not discuss it.

> **Thm** (Jordan Curve Theorem)
> A simple closed curve in $\mathbb{R}^2$ divides $\mathbb{R}^2$ into 2 path-connected components, with one bounded and one unbounded.

Remark: "closed curve" means continuous curve $\vec{x}(t)$, $a \le t \le b$ with $\vec{x}(a) = \vec{x}(b)$. And one can show that it is a "closed subset" in $\mathbb{R}^2$.

eg: