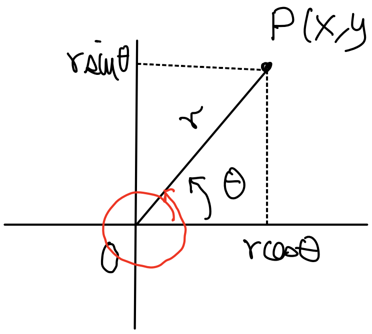
Polar Coordinates in $\mathbb{R}^2$

$P = (x, y) \in \mathbb{R}^2$ can be represented by

$(r\cos\theta, r\sin\theta)$

where $r = \sqrt{x^2+y^2}$ = distance from origin

θ = angle from the positive x-axis to $\overrightarrow{OP}$ in counter-clockwise direction



Remarks: (i) $(r\cos\theta, r\sin\theta) = (r\cos(\theta+2k\pi), r\sin(\theta+2k\pi))$

for any $k \in \mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$

(ii) For $P = (0,0)$, then $\begin{cases} r = 0 \\ \theta \text{ is not well-defined} \end{cases}$

(iii) For our defn, we usually set

- $r \in [0, \infty)$ $\quad (r \geq 0)$
- $\theta \in [0, 2\pi)$ $\quad (0 \leq \theta < 2\pi)$

But in some book, $\begin{cases} r \in \mathbb{R} \text{ (can be negative as in Textbook) (see later examples)} \\ \theta \in \mathbb{R} \end{cases}$

Change of Coordinates formula

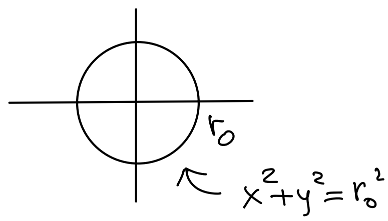
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \& \quad \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{cases} \quad (\text{for } x>0)$$

Curves in Polar Coordinates

eg : Circle of radius $r_0 > 0$, centered at origin

Polar Equation :

$$r = r_0 .$$



Parametric Form (in polar)

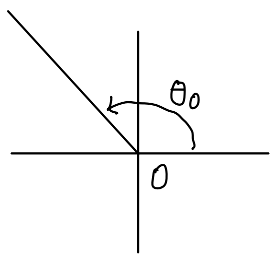
$$\begin{cases} r = r_0 \\ \theta = t , \quad t \in [0, 2\pi] \end{cases}$$

$$\left(\vec{x}(t) = (r_0\cos t, r_0\sin t) \right)$$

eg Half ray from origin

Polar Equation

$$\theta = \theta_0$$



Parametric form (in polar)

$$\begin{cases} r = t , \quad t \in [0, \infty) \\ \theta = \theta_0 \end{cases}$$

eg: Archimedes Spiral

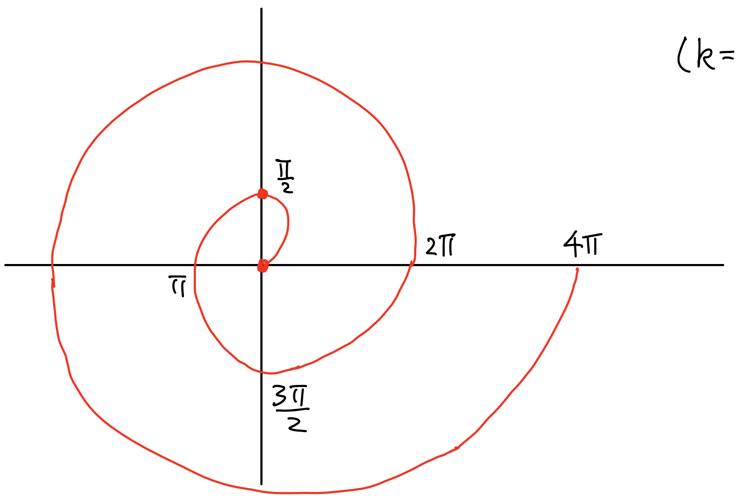
$k>0$ is a constant.

Polar Equation

$r=k\theta$

Parametric form (in pole)

$\begin{cases} r=kt \\ \theta=t \end{cases}$ $\quad t\in[0,\infty)$

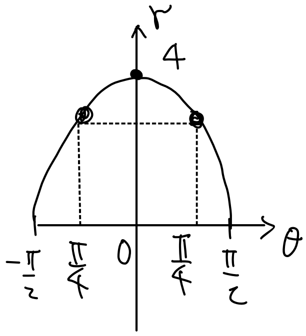


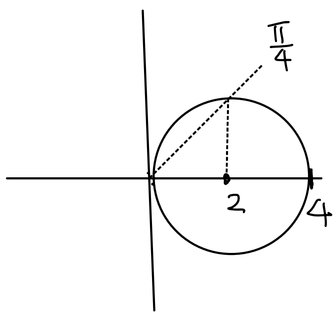
Remark: $\vec{x}(t)=(r\cos\theta, r\sin\theta)=(kt\cos t, kt\sin t)=k(t\cos t, t\sin t)$

$\Rightarrow \vec{x}'(t)=k(\cos t-t\sin t, \sin t+t\cos t)$ is the tangent vector at $\vec{x}(t)=k(t\cos t, t\sin t)$.

($(r'(t), \theta'(t))=(k,1)$ is not the tangent vector in $\mathbb{R}^2$)

eg: $r=4\cos\theta$ $\left(\geq 0, -\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}\right)$





$r=4\cos\theta$

$\Rightarrow r^2=4r\cos\theta$

i.e. $x^2+y^2=4x$

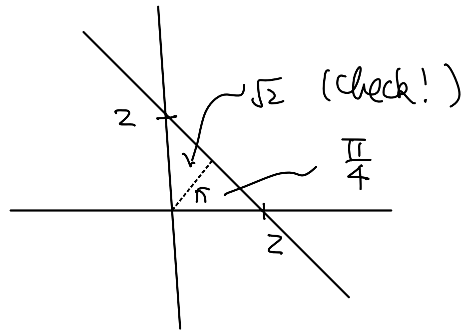
eg: $r\cos(\theta - \frac{\pi}{4}) = \sqrt{2}$

$$\Downarrow$$

$$r(\cos\theta\cos\tfrac{\pi}{4} + \sin\theta\sin\tfrac{\pi}{4}) = \sqrt{2}$$

$$\Rightarrow \quad \frac{r\cos\theta}{\sqrt{2}} + \frac{r\sin\theta}{\sqrt{2}} = \sqrt{2}$$

$$\Rightarrow \quad x + y = 2$$



Negative r

Our convention is $r \geq 0$.
But sometimes is convenient to allow $r < 0$ by the interpretation

$$\begin{aligned}(x, y) &= (r\cos\theta, r\sin\theta) \\ &= (-|r|\cos\theta, -|r|\sin\theta) \\ &= -(|r|\cos\theta, |r|\sin\theta) \qquad \big(= (|r|\cos(\theta+\pi), |r|\sin(\theta+\pi))\big)\end{aligned}$$

eg: $r = -2$, $\theta = \frac{\pi}{6}$ $\quad (x, y) = (-2\cos\frac{\pi}{6}, -2\sin\frac{\pi}{6}) = -(\sqrt{3}, 1) = (-\sqrt{3}, -1)$

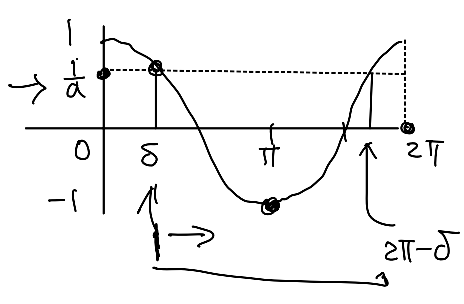
eg: $r = 1 - (1+\varepsilon)\cos\theta$, $\quad \varepsilon > 0$

$\quad = 1 - a\cos\theta$, $\quad a = 1 + \varepsilon > 1$

Soln:

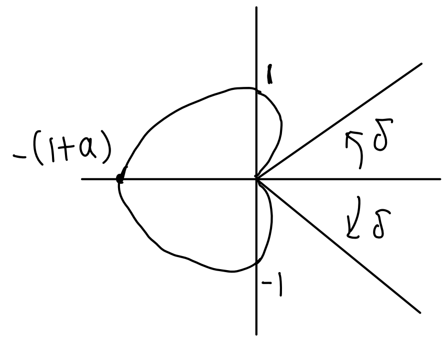
Case 1 $\quad r \geq 0$

$\Rightarrow \quad 1 - a\cos\theta \geq 0 \Rightarrow \cos\theta \leq \frac{1}{a} < 1$



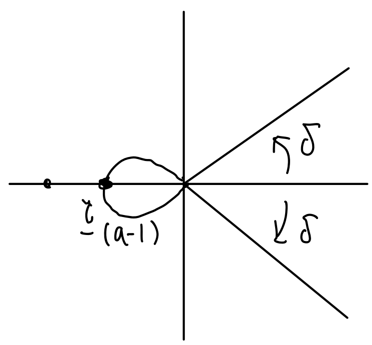
$\Rightarrow$ θ cannot run through the whole interval $[0, 2\pi]$

but only $[\delta, 2\pi-\delta]$ where $\delta = \cos^{-1}(\frac{1}{a})$.



$r = 1 - a\cos\theta$

<u>Case 2</u> $r < 0$, $0 \le \theta < \delta$ & $2\pi - \delta < \theta \le 2\pi$

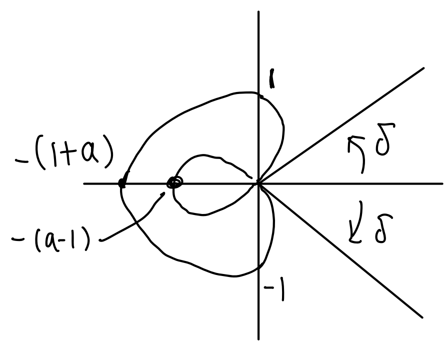


$r = 1 - a\cos\theta$

$|r| = a\cos\theta - 1$

$|r| = a - 1$ for $\theta = 0$ & 2π

So if we allow $r \in \mathbb{R}$, then $r = 1 - a\cos\theta$ can be defined for all $\theta \in [0, 2\pi]$ & the curve becomes a curve with <u>self-intersection</u>:



Coordinates Systems in $\mathbb{R}^3$ (generalizing polar coordinates)

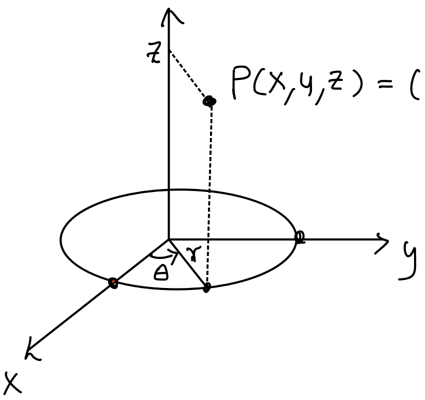
Cylindrical Coordinates

$(x, y, z) \longrightarrow (r, \theta, z)$

where $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ i.e. (r, θ) is polar coordinates for xy-plane.

Formula

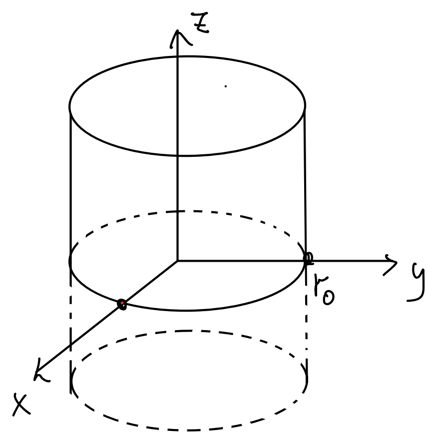
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$



$P(x, y, z) = (r\cos\theta, r\sin\theta, z)$

↘ cylindrical coordinates (r, θ, z)

eg: $r = r_0$ constant is the equation for the cylinder in the figure.



eg Helix $(a\cos t, a\sin t, bt)$ $(a > 0,\ b \in \mathbb{R})$

can be represented in cylindrical coordinates by

$$\begin{cases} r = a \\ \theta = t \\ z = bt \end{cases} \qquad t \in [0, 2\pi]$$

(Ex: Sketch the curve)

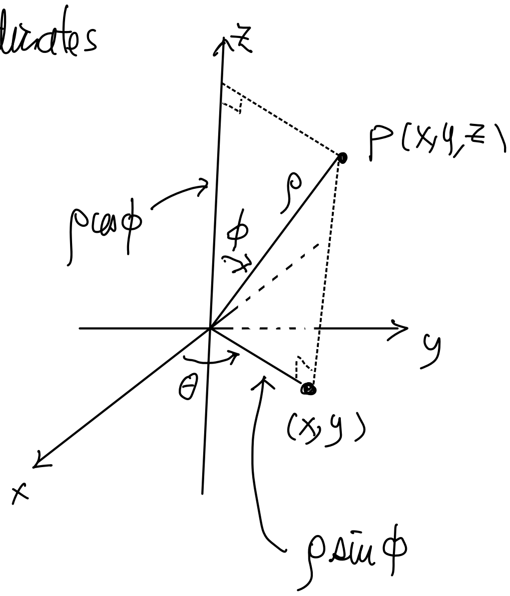
Spherical Coordinates

$P = (x,y,z) \in \mathbb{R}^3$ can be represented by

$\rho =$ distance from origin $= \sqrt{x^2+y^2+z^2}$

$\theta = \theta$ as in cylindrical coordinates

$\phi =$ angle from positive z-axis to $\overrightarrow{OP}$.



Remark: $\phi \in [0,\pi]$

Formulae

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho\cos\phi \end{cases}$$

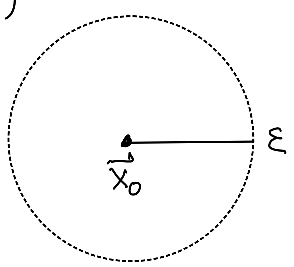
(Tutorial for Egs)

Topological Terminology in $\mathbb{R}^n$

Def
- $B_\varepsilon(\vec{x}_0) = \{ \vec{x} \in \mathbb{R}^n : \| \vec{x} - \vec{x}_0 \| < \varepsilon \}$ is called the <u>open ball</u> of radius ε and centered at $\vec{x}_0$
- $\overline{B_\varepsilon(\vec{x}_0)} = \{ \vec{x} \in \mathbb{R}^n : \| \vec{x} - \vec{x}_0 \| \leq \varepsilon \}$ is called the <u>closed ball</u> of radius ε and centered at $\vec{x}_0$

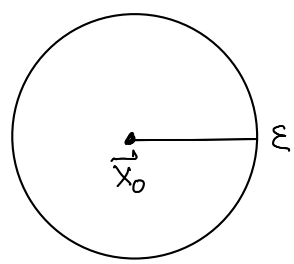
<u>Remark</u>: If $n=2$, $B_\varepsilon(\vec{x}_0)$, $\overline{B_\varepsilon(\vec{x}_0)}$ are referred as <u>open disk</u>, <u>closed disk</u> and denoted by $D_\varepsilon(\vec{x}_0)$, $\overline{D_\varepsilon(\vec{x}_0)}$ in some textbooks.

$B_\varepsilon(\vec{x}_0)$



points on the dotted "line" are not included

$\overline{B_\varepsilon(\vec{x}_0)}$



points on the solid "line" are included

<u>Recall notation</u>: " $\exists$ " : there exist(s)

" $\forall$ " : for all

Def: Let S be a set in $\mathbb{R}^n$.

(1) The <u>interior of S</u> is the set

$$\operatorname{Int}(S) = \{ \vec{x} \in \mathbb{R}^n : \exists\, \varepsilon > 0 \text{ s.t. } B_\varepsilon(\vec{x}) \subset S \}$$

Points in $\operatorname{Int}(S)$ are called <u>interior points</u> of S

(2) The <u>exterior of S</u> is the set

$$\operatorname{Ext}(S) = \{ \vec{x} \in \mathbb{R}^n : \exists\, \varepsilon > 0 \text{ s.t. } B_\varepsilon(\vec{x}) \subset \mathbb{R}^n \setminus S \}$$

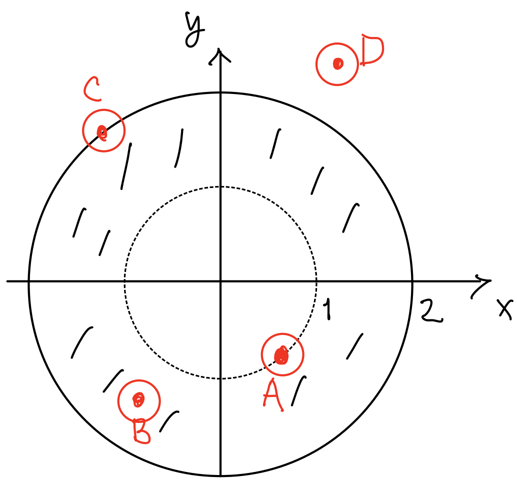
Points in $\operatorname{Ext}(S)$ are called <u>exterior points</u> of S

(3) The <u>boundary of S</u> is the set

$$\partial S = \left\{ \vec{x} \in \mathbb{R}^n : \forall\, \varepsilon > 0 \text{ s.t. } \begin{matrix} B_\varepsilon(\vec{x}) \cap S \neq \phi, \ \& \\ B_\varepsilon(\vec{x}) \cap (\mathbb{R}^n \setminus S) \neq \phi \end{matrix} \right\}$$

Points on ∂S are called <u>boundary points</u> of S

<u>eg</u>: $S = \{ (x,y) \in \mathbb{R}^2 : 1 < x^2 + y^2 \leq 4 \} \subset \mathbb{R}^2$



A = boundary point C = boundary point

B = interior point D = exterior point

$$\operatorname{Int}(S) = \{ (x,y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4 \}$$

$$\operatorname{Ext}(S) = \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1 \text{ or } x^2 + y^2 > 4 \}$$

$$\partial S = \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \text{ or } x^2 + y^2 = 4 \}$$

Prop Let $S \subset \mathbb{R}^n$. Then

(1) $\mathbb{R}^n$ = disjoint union of $\text{Int}(S)$, $\text{Ext}(S)$ and ∂S

(2) $\text{Int}(S) \subseteq S$

$\text{Ext}(S) \subseteq \mathbb{R}^n \setminus S$

(3) A point on ∂S may or may not be in S

(For statement (3), see points A & C in the above eg.)

Def A set $S \subset \mathbb{R}^n$ is called

(1) open if $\forall \vec{x} \in S$, $\exists \varepsilon > 0$ such that $B_\varepsilon(\vec{x}) \subseteq S$

(2) closed if $\mathbb{R}^n \setminus S$ is open

Equivalent definition:

(1) S open $\Leftrightarrow$ $S = \text{Int}(S)$

(2) S closed $\Leftrightarrow$ $S = \text{Int}(S) \cup \partial S$

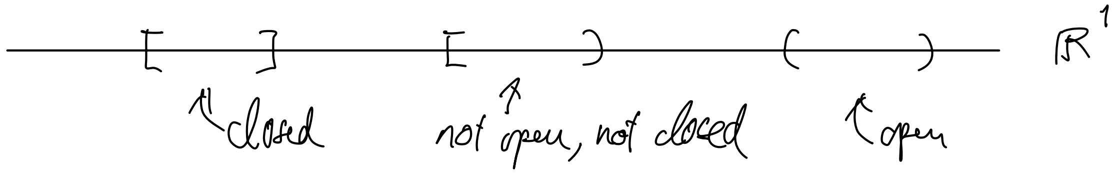
(check!)

eg Is $S = \{(x,y) \in \mathbb{R}^2 : 1 < x^2 + y^2 \leq 4\}$ open or closed?

Answer: Not open, and

Not closed!

(Similar to



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