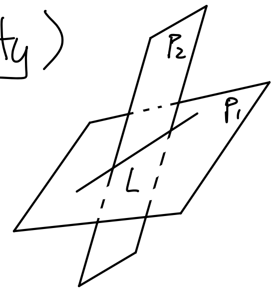
eg: <u>Line in $\mathbb{R}^3$ by equations</u>

Two planes intersect at a line (if not empty)



$$\begin{cases} x+y+6z=6 \\ x-y-2z=-2 \end{cases} \qquad \left(\begin{array}{c} (1,1,6) \ \& \ (1,-1,-2) \\ \text{are linearly indep.} \end{array} \right)$$

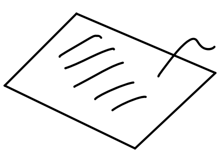
is a line. Then Gaussian Elimination will give us a parametric form of the line. i.e solving the system of linear equation by setting a variable to be a parameter: eg

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -4 \\ 1 \end{bmatrix} \quad (\text{by setting } z=t) \quad \begin{array}{c} (\text{Ex!}) \\ (\text{linear algebra!}) \end{array}$$

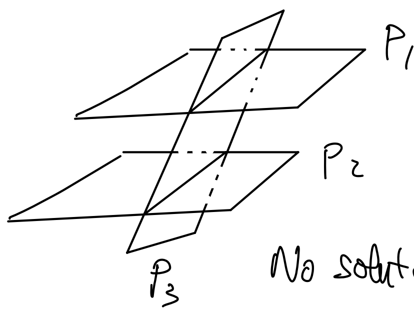
eg: How about 3 linear equations?

Linear Alg $\Rightarrow$
- Case 1: unique solution, i.e. intersection = {point}.
- Case 2: Infinitely many solutions; could be a line or a plane
- Case 3: No solution, i.e. no intersection.

eg.



infinitely many solutions



No solution

(Ex: Try other situations)

Remark: In n dim., a (hyper)plane is given by $\underline{\vec{X}\cdot\vec{n} = c}$.

as in planes in $\mathbb{R}^3$ (dim (hyperplane) $= n-1$)

Then linear algebra $\Rightarrow$ all possible situations for intersections of (hyper)planes.

(Discussion omitted)

Curves in $\mathbb{R}^n$

Defn: Let $I \subset \mathbb{R}$ be an <u>interval</u>. A (continuous) <u>curve</u> in $\mathbb{R}^n$ is a <u>continuous (vector-valued) function</u>

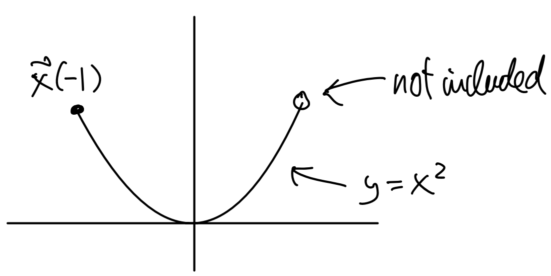
$$\vec{X} : I \to \mathbb{R}^n$$

i.e. $t \in I$, $\vec{X}(t) = (X_1(t), \cdots, X_n(t)) \in \mathbb{R}^n$ such that every <u>component function</u> $X_i(t)$ is continuous ($i = 1, \cdots, n$)

<u>egs</u> (i) $\vec{X} : [-1, 1) \to \mathbb{R}^2$

$\vec{X}(t) = (t, t^2)$

$[X = t, y = t^2 \Rightarrow y = X^2]$



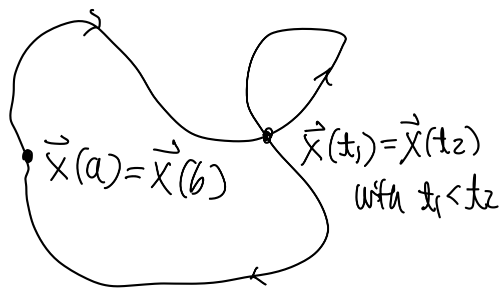
(ii) Of course, parametric form of a line gives a "curve" $\vec{X}(t) = \vec{p} + t\vec{q}$, $t \in (-\infty, \infty)$

Defn: A curve $\vec{x} : [a,b] \to \mathbb{R}^n$ is said to be

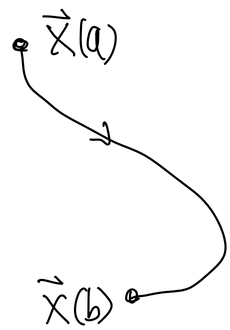
(i) <u>closed</u> if $\vec{x}(a) = \vec{x}(b)$

(ii) <u>simple</u> if $\vec{x}(t_1) \neq \vec{x}(t_2)$ for $a \leq t_1 < t_2 \leq b$ except possibly at $t_1 = a$ & $t_2 = b$.

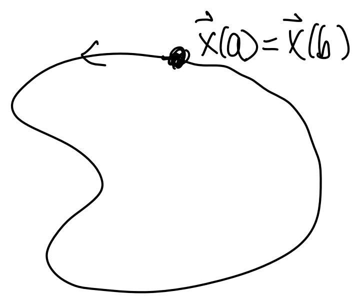
<u>eg</u>:



<u>closed</u>, <u>not simple</u>



<u>not closed</u>, <u>simple</u>



<u>Closed & simple</u>

(simple closed curve)

Thm: Let $\vec{x}(t) = (x_1(t), \cdots, x_n(t))$. Then

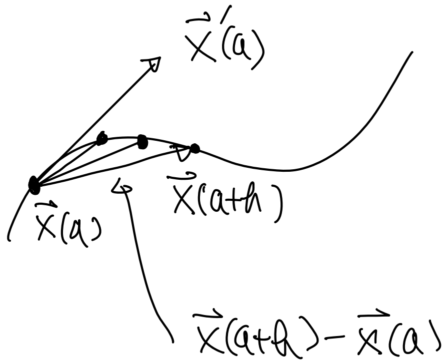
(1) $$\lim_{t \to a} \vec{x}(t) = \left(\lim_{t \to a} x_1(t), \cdots, \lim_{t \to a} x_n(t) \right)$$

(2) $$\vec{x}'(t) \overset{\text{def}}{=} \lim_{h \to 0} \frac{\vec{x}(t+h) - \vec{x}(t)}{h} = (x_1'(t), \cdots, x_n'(t))$$

(provided limits exist)

Defn : $\vec{x}'(a)$ = tangent vector of $\vec{x}(t)$ at $t=a$.

Picture :



Physics : If $\vec{x}(t)$ = displacement at time t

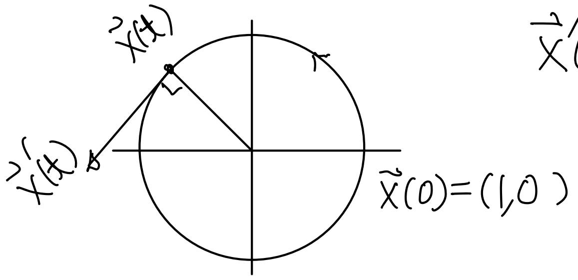
Then $\vec{x}'(t)$ = velocity (vector) at time t

$\vec{x}''(t)$ = acceleration (vector)

$\|\vec{x}'(t)\|$ = speed.

eg: $\vec{x}(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$

($\overset{\shortparallel}{x}$ $\overset{\shortparallel}{y}$ $\Rightarrow x^2+y^2=1$ the unit circle)



(simple closed)

$\vec{x}'(t) = (-\sin t, \cos t)$ is the tangent vector

[$\vec{v}$ = velocity $= \vec{x}'(t) = (-\sin t, \cos t)$

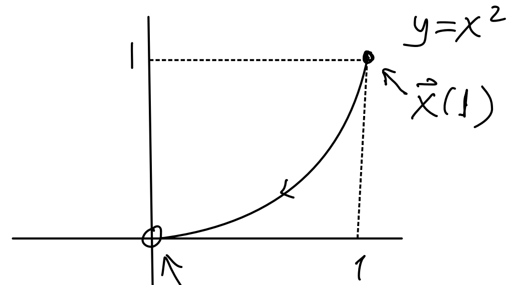
$\vec{a}$ = acceleration $= \vec{x}''(t) = (-\cos t, -\sin t)$
$= -\vec{x}(t)$

speed $= \|\vec{x}'(t)\| = 1$]

eg $\vec{x} : [1, \infty) \to \mathbb{R}^2$

$\vec{x}(t) = \left(\frac{1}{t}, \frac{1}{t^2}\right)$

($\overset{\shortparallel}{x}$ $\overset{\shortparallel}{y}$ $\Rightarrow y = x^2$)



$\lim_{t\to\infty} \vec{x}(t) = (0,0)$
(not included in the curve)

Rules

Let $\vec{x}(t), \vec{y}(t)$ be curves in $\mathbb{R}^n$, $c \in \mathbb{R}$ be a constant

$f(t)$ be a real-valued function. Then

(1) $(\vec{x}(t)+\vec{y}(t))' = \vec{x}'(t)+\vec{y}'(t)$

(2) $(c\vec{x}(t))' = c\,\vec{x}'(t)$

(3) $(f(t)\vec{x}(t))' = f'(t)\,\vec{x}(t) + f(t)\,\vec{x}'(t)$

(4) $(\vec{x}(t)\cdot\vec{y}(t))' = \vec{x}'(t)\cdot\vec{y}(t) + \vec{x}(t)\cdot\vec{y}'(t)$

(5) For $n=3$,

$$(\vec{x}(t)\times\vec{y}(t))' = \vec{x}'(t)\times\vec{y}(t) + \vec{x}(t)\times\vec{y}'(t)$$

<u>Remark</u>: (3), (4) & (5) are all called <u>product rules</u>.

Arclength (of a curve)

Let $\vec{x}(t)$ be a curve with $\vec{x}'(t)$ exists and continuous

<u>Def</u>: <u>Arclength</u> of $\vec{x}(t)$ for $a \leq t \leq b$ is

$$S = \int_a^b \|\vec{x}'(t)\|\,dt$$

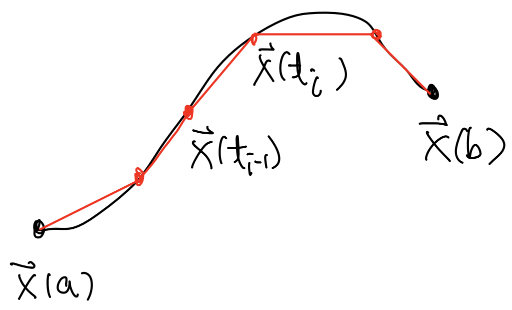
<u>Remark</u>: If $\vec{x}(t)$ = displacement at time t

then $\vec{x}'(t)$ = velocity

$\|\vec{x}'(t)\|$ = speed

$\int_a^b \|\vec{x}'(t)\|\,dt$ = distance travelled.

Idea of the defn (from mathematician's point of view)



approximate the curve by straight line segments.

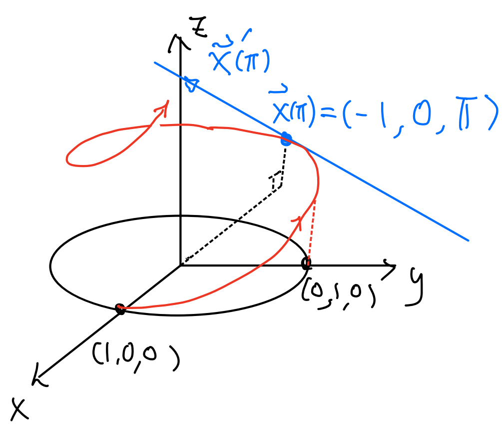
$$s \approx \sum_i \|\vec{x}(t_i) - \vec{x}(t_{i-1})\| \qquad \left(\vec{x}'(t_i) = \lim_{t \to t_i} \frac{\vec{x}(t) - \vec{x}(t_i)}{t - t_i} \right)$$

$$\approx \sum_i \|\vec{x}'(t_i)\| (t_i - t_{i-1}) \longrightarrow \int_a^b \|\vec{x}'(t)\| \, dt$$

"as the approximation get better & better"

eg: (Helix)

$\vec{x}(t) = (\cos t, \sin t, t)$ (curve in $\mathbb{R}^3$) $t \in [0, 2\pi]$



(a) Find the tangent line of $\vec{x}$ at $t = \pi$

(b) Find arclength of the helix

Soln (a): $\vec{x}(t) = (\cos t, \sin t, t)$

$\vec{x}'(t) = (-\sin t, \cos t, 1)$

$\therefore$ $\vec{x}'(\pi) = (0, -1, 1)$ is the tangent vector of $\vec{x}$ at $t = \pi$

(Of course $\vec{x}(\pi) = (-1, 0, \pi)$ is a point on the tangent line.)

$\therefore$ The tangent line at $t = \pi$ is given by

$$\vec{y}(\tau) = \vec{x}(\pi) + \tau \vec{x}'(\pi)$$

$$= (-1, 0, \pi) + \tau(0, -1, 1) \qquad (\tau \in \mathbb{R})$$

(b) $\|\vec{x}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2}$

$\Rightarrow$ arclength $s = \int_0^{2\pi} \|\vec{x}'(t)\| \, dt = \int_0^{2\pi} \sqrt{2} \, dt$

$$= 2\sqrt{2}\pi.$$

<u>Remark</u>: Arclength is independent of change of parameters! (Proof omitted)
(Ex)