eg (revisit) $\vec{a} = (2,3,5)$, $\vec{b} = (1,2,3)$. Find $\vec{a}\times\vec{b}$.

Soln: $\vec{a} = 2\hat{i} + 3\hat{j} + 5\hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$

$$\begin{aligned}\vec{a}\times\vec{b} &= (2\hat{i} + 3\hat{j} + 5\hat{k}) \times (\hat{i} + 2\hat{j} + 3\hat{k}) \\ &= \vec{0} + 3(-\hat{k}) + 5\hat{j} + 4\hat{k} + \vec{0} + 10(-\hat{i}) \\ &\qquad + 6(-\hat{j}) + 9\hat{i} + \vec{0} \\ &= -\hat{i} - \hat{j} + \hat{k} \quad ※\end{aligned}$$

Triple Product (only in $\mathbb{R}^3$)

Let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$

The triple product of $\vec{a}$, $\vec{b}$ & $\vec{c}$ (order is important) is defined by $(\vec{a}\times\vec{b})\cdot\vec{c}$

Note: Assume $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$ & $\vec{c} = (c_1, c_2, c_3)$

$$\begin{aligned}(\vec{a}\times\vec{b})\cdot\vec{c} &= \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right) \cdot (c_1, c_2, c_3) \\ &= c_1\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\end{aligned}$$

$$\therefore \boxed{(\vec{a}\times\vec{b})\cdot\vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}} \quad \left(\text{expansion formula along } 3^{rd} \text{ row}\right)$$

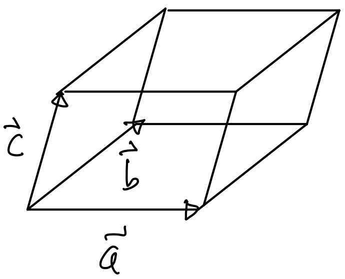
Remark: It is easy to obtain

$$(\vec{a}\times\vec{b})\cdot\vec{c} = (\vec{b}\times\vec{c})\cdot\vec{a} = (\vec{c}\times\vec{a})\cdot\vec{b} \qquad (\text{Ex})$$

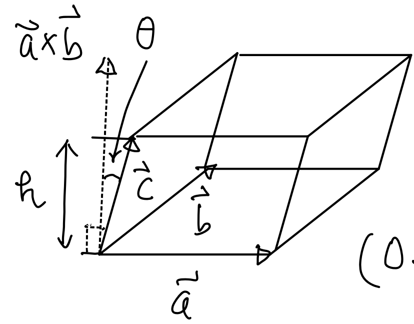
$$= -(\vec{b}\times\vec{a})\cdot\vec{c} = -(\vec{a}\times\vec{c})\cdot\vec{b} = -(\vec{c}\times\vec{b})\cdot\vec{a}$$

Geometric meaning

$\left|(\vec{a}\times\vec{b})\cdot\vec{c}\right|$ = Volume of the parallelopiped spanned by $\vec{a}, \vec{b}$ & $\vec{c}$.



Pf:



$(0\leq\theta<\frac{\pi}{2})$

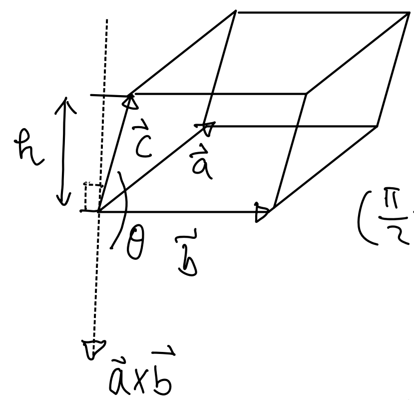
$h = \|\vec{c}\|\cos\theta$

and $(\vec{a}\times\vec{b})\cdot\vec{c} = \|\vec{a}\times\vec{b}\|\,\|\vec{c}\|\cos\theta$

$= \text{Area}(\text{parallelogram spanned by } \vec{a}, \vec{b}) \cdot h$

= Volume of the parallelopiped.

Ex: For case $\frac{\pi}{2}<\theta\leq\pi$, the proof similar.



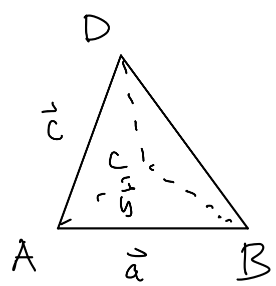
$(\frac{\pi}{2}<\theta\leq\pi)$

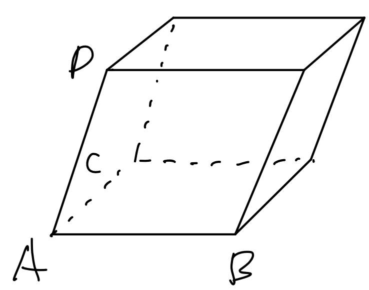
$\#$

Remarks: (i) $(\vec{a}\times\vec{b})\cdot\vec{c}=0 \Leftrightarrow \text{Vol(parallelepiped)}=0$

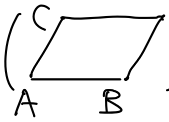
$\Leftrightarrow \{\vec{a},\vec{b},\vec{c}\}$ are linearly dependent.

(ii) Tetrahedron





Vol (Tetrahedron)

$$= \frac{1}{3} \text{Area}(\triangle ABC) \cdot \text{height} = \frac{1}{3} \cdot \frac{1}{2} \text{Area}\Big($$  $$\Big) \cdot \text{height}$$

$$= \frac{1}{6} \text{Vol (Parallelepiped)}$$

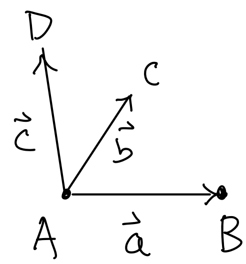
$$= \frac{1}{6} \left| (\vec{a} \times \vec{b}) \cdot \vec{c} \right|$$

<u>eg</u> Let $A = (1,0,1)$, $B = (1,1,2)$, $C = (2,1,1)$, $D = (2,1,3)$

Find volume of tetrahedron $ABCD$

<u>Soln</u>: $\vec{AB} = (1,1,2) - (1,0,1)$

$\quad = (0,1,1)$



$\vec{AC} = (2,1,1) - (1,0,1) = (1,1,0)$

$\vec{AD} = (2,1,3) - (1,0,1) = (1,1,2)$

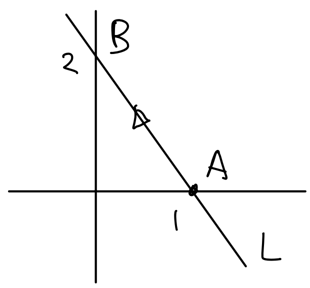
$$\text{Vol (Tetrahedron)} = \frac{1}{6} \left| (\vec{AB} \times \vec{AC}) \cdot \vec{AD} \right|$$

$$= \frac{1}{6} \left| \det \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \right| = \frac{1}{6} |-2| = \frac{1}{3} \quad . \#$$

Linear Objects in $\mathbb{R}^n$

(lines, plane, k-plane, hyperplane)

Line eg in $\mathbb{R}^2$



Equation form: $2x + y = 2$

Parametric form:

$$(x,y) = \overrightarrow{OA} + t\,\overrightarrow{AB}, \quad t \in \mathbb{R}$$

$$= (1,0) + t\big((0,2) - (1,0)\big)$$

$$= (1-t,\ 2t)$$

i.e. $\begin{cases} x = 1 - t \\ y = 2t \end{cases}$, $t \in \mathbb{R}$

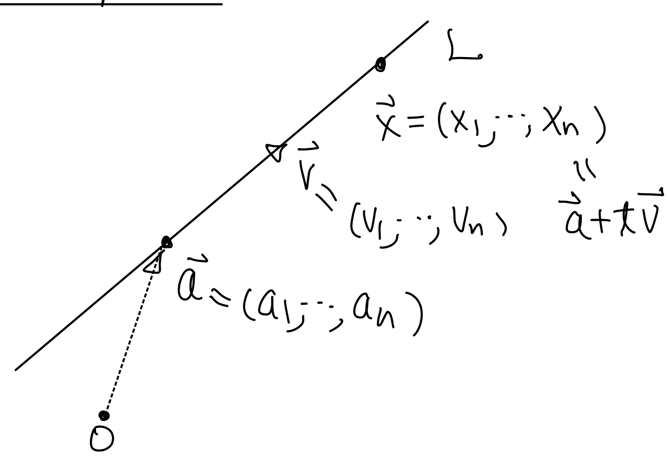
Note: Symmetric / Slope form $\quad \dfrac{x-1}{-1} = \dfrac{y-0}{2}$ (Ex!)

Parametric Form of a line in $\mathbb{R}^n$ ($n=3$ particular)

Let L = a line in $\mathbb{R}^n$

$\vec{a}$ = a point on L

$\vec{v}$ = a direction vector of L $(\vec{v} \neq \vec{0})$



Then

parametric form of L $\vec{x} = \vec{a} + t\,\vec{v}$ $t \in \mathbb{R}$ called a parameter

(L is parametrized by $t \in \mathbb{R}$)

i.e. $(x_1, \cdots, x_n) = (a_1, \cdots, a_n) + t(v_1, \cdots, v_n)$

$= (a_1 + t v_1, \cdots, a_n + t v_n)$

i.e. $\begin{cases} x_1 = a_1 + t v_1 \\ \vdots \\ x_n = a_n + t v_n \end{cases} \quad t \in \mathbb{R}$

<u>eg</u> A line L in $\mathbb{R}^3$ passes through

$A = (1,2,3)$, $B = (-1,3,5)$

<u>Soln</u>: $\left(\begin{array}{l} \text{Choose } \vec{a} = A \text{ (or } B\text{) as vector,} \\ \vec{v} = \overrightarrow{AB} \text{ (or } \overrightarrow{BA}\text{)} \end{array} \right)$

A parametrization of L is $\vec{x} = (1,2,3) + t((-1,3,5) - (1,2,3))$

$= (1,2,3) + t(-2,1,2)$ ※

(In high school notations: $x = 1 - 2t$, $y = 2 + t$, $z = 3 + 2t$)

<u>Remarks</u>: (i) Parametric form is not unique (many choice as in eg)

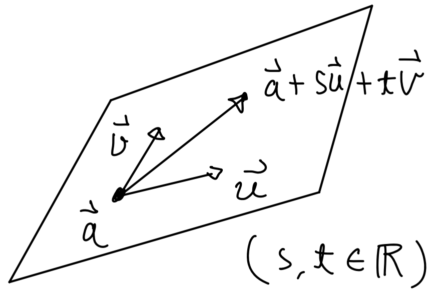
(ii) From (※), we get symmetric form:

$$\frac{x-1}{-2} = \frac{y-2}{1} = \frac{z-3}{2} \quad (= t)$$

$$\Leftrightarrow \begin{cases} x - 1 = -2(y-2) \\ 2(y-2) = z - 3 \end{cases}$$

Planes in $\mathbb{R}^3$

(1) P = a plane in $\mathbb{R}^3$



$(s, t \in \mathbb{R})$

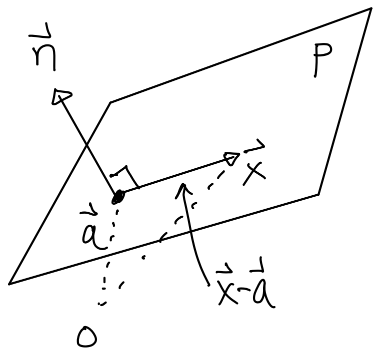
$\vec{a}$ = a point on P

$\vec{u}, \vec{v}$ = 2 linearly independent vectors on P.

Then

$$\boxed{\begin{array}{c}\underline{\text{Parametric Form of } P} \\ \vec{x} = \vec{a} + s\vec{u} + t\vec{v} \\ \text{two parameters}\end{array}}$$

(2) P = a plane in $\mathbb{R}^3$



$\vec{a}$ = a point on P

$\vec{n}$ = a normal vector of P

(i.e. $\vec{n}$ is perpendicular to P & $\vec{n} \neq \vec{0}$)

Let $\vec{a} = (a_1, a_2, a_3)$, $\vec{n} = (n_1, n_2, n_3)$ and $\vec{x} = (x, y, z)$

$$\vec{x} \in P \iff (\vec{x}-\vec{a}) \perp \vec{n}$$

$$\iff (\vec{x}-\vec{a}) \cdot \vec{n} = 0 \quad (\iff \vec{x}\cdot\vec{n} = \vec{a}\cdot\vec{n})$$

$$\iff (x-a_1, y-a_2, z-a_3) \cdot (n_1, n_2, n_3) = 0$$

$$\iff n_1x + n_2y + n_3z = \underbrace{n_1a_1 + n_2a_2 + n_3a_3}_{\text{constant}}.$$

Equation of P (general in $\mathbb{R}^3$)

$$n_1 x + n_2 y + n_3 z = c$$

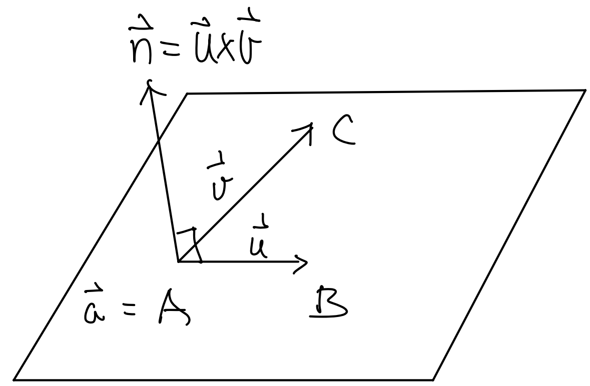
$(c = \vec{a} \cdot \vec{n})$
provided $\vec{n} = (n_1, n_2, n_3) \neq \vec{0}$.

eg: Suppose P is a plane (in $\mathbb{R}^3$) passing through

$A = (0,0,1)$, $B = (0,2,0)$, $C = (-1,1,0)$

Represent P using (i) parametric form; (ii) equation.

Soln: (i) $\left(\begin{array}{l} \text{Pick } \vec{a} = A, B \text{ or } C \\ \vec{u}, \vec{v} = \vec{AB}, \vec{AC} \text{ or } \cdots \end{array} \right)$



$\vec{u} = \vec{AB} = (0,2,0) - (0,0,1) = (0,2,-1)$

$\vec{v} = \vec{AC} = (-1,1,0) - (0,0,1) = (-1,1,-1)$

Then the parametric form of P

is $\quad \vec{x} = (0,0,1) + s(0,2,-1) + t(-1,1,-1) \quad (s, t \in \mathbb{R})$

(ii) Take $\vec{n} = \vec{u} \times \vec{v}$ $\quad$ ($\vec{u}, \vec{v}$ are as in (i))

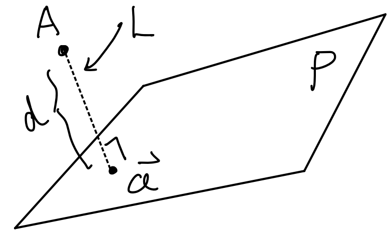
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & -1 \\ -1 & 1 & -1 \end{vmatrix} = (-1, 1, 2) \qquad \text{(Check!)}$$

$\Rightarrow$ Equation of P: $\quad \big((x,y,z) - (0,0,1)\big) \cdot (-1,1,2) = 0$

i.e. $\quad -x + y + 2z = 2 \qquad$ (check!)

$\#$

eg: Find the distance between $A=(2,1,1)$ and the $P: -x+2y-z=-4$ — (*)



Soln: From (*), $\vec{n}=(-1,2,-1) \perp P$

Consider the line L (passing A in the direction of $\vec{n}$)

$$\vec{X}=\vec{A}+t\vec{n}$$
$$=(2,1,1)+t(-1,2,-1)$$
$$=(2-t,\ 1+2t,\ 1-t) \qquad t\in\mathbb{R}$$

Let $\vec{a}$ be the intersection of L and P, then $\vec{a}$ can be found as follows:

put $(x,y,z)=(2-t,\ 1+2t,\ 1-t)$ into eqt. (*) of P.

$$-(2-t)+2(1+2t)-(1-t)=-4$$
$$\Rightarrow \quad t=-\tfrac{1}{2} \quad \text{(Check!)}$$

& $\vec{a}=\left(2-(-\tfrac{1}{2}),\ 1+2(-\tfrac{1}{2}),\ 1-(-\tfrac{1}{2})\right)=\left(\tfrac{5}{2},0,\tfrac{3}{2}\right)$ (check!)

Hence

distance between A & P = distance between A & $\vec{a}$

$$=\sqrt{\left(2-\tfrac{5}{2}\right)^2+(1-0)^2+\left(1-\tfrac{3}{2}\right)^2}$$
$$=\frac{\sqrt{6}}{2} \quad \text{(Check!)}$$

※