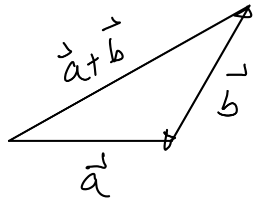
Triangle Inequality

Let $\vec{a}, \vec{b} \in \mathbb{R}^n$. Then

$$\|\vec{a}+\vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$$



Equality holds $\Leftrightarrow$ $\vec{a} = r\vec{b}$ or $\vec{b} = r\vec{a}$ for some $r \geq 0$

Pf: $\|\vec{a}+\vec{b}\|^2 = (\vec{a}+\vec{b}) \cdot (\vec{a}+\vec{b}) = \|\vec{a}\|^2 + 2\vec{a}\cdot\vec{b} + \|\vec{b}\|^2$

Cauchy-Schwarz $\leq \|\vec{a}\|^2 + 2\|\vec{a}\|\|\vec{b}\| + \|\vec{b}\|^2$

$= (\|\vec{a}\| + \|\vec{b}\|)^2$

$\Rightarrow \quad \|\vec{a}+\vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$

Equality holds $\Leftrightarrow$ $\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|$ $\Rightarrow$ Equality holds for Cauchy-Schwarz

$\Rightarrow$ $\vec{a} = r\vec{b}$ or $\vec{b} = r\vec{a}$ for some $r \in \mathbb{R}$

Putting back $\Rightarrow$ $\left.\begin{array}{l} r\|\vec{b}\|^2 = \|\vec{a}\|\|\vec{b}\| \\ \text{or } r\|\vec{a}\|^2 = \|\vec{a}\|\|\vec{b}\| \end{array}\right\} \Rightarrow r \geq 0$ provided $\vec{a} \neq 0$ or $\vec{b} \neq 0$

If $\vec{a} = \vec{b} = \vec{0}$, the statement is trivially correct. #

Option Ex: Cauchy-Schwarz inequality $\Leftrightarrow$ Triangle Inequality.

("$\Rightarrow$" done, "$\Leftarrow$" Ex.)

Special structure of $\mathbb{R}^3$: Cross Product $\vec{a} \times \vec{b}$

Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$

Then the cross product $\vec{a} \times \vec{b}$ is defined by

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{k}$$

$$= \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right)$$

where $\hat{i} = (1,0,0)$, $\hat{j} = (0,1,0)$, $\hat{k} = (0,0,1)$.

eg: Let $\vec{a} = (2,3,5)$ & $\vec{b} = (1,2,3)$

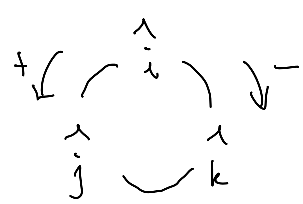
$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 5 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 2 & 3 \end{vmatrix} \hat{i} - \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} \hat{j} + \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} \hat{k} = -\hat{i} - \hat{j} + \hat{k}$$

$$= (-1, -1, 1)$$

Remark

(check!)

$$\boxed{\begin{array}{lll} \hat{i} \times \hat{i} = \vec{0} & \hat{i} \times \hat{j} = \hat{k} & \hat{i} \times \hat{k} = -\hat{j} \\ \hat{j} \times \hat{i} = -\hat{k} & \hat{j} \times \hat{j} = \vec{0} & \hat{j} \times \hat{k} = \hat{i} \\ \hat{k} \times \hat{i} = \hat{j} & \hat{k} \times \hat{j} = -\hat{i} & \hat{k} \times \hat{k} = \vec{0} \end{array}}$$



Properties of Cross Product

Let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$, $\alpha, \beta \in \mathbb{R}$

Algebraic ((1) & (2) follow from properties of determinant)

(1) $\vec{a} \times \vec{b} = - \vec{b} \times \vec{a}$

(2) $(\alpha \vec{a} + \beta \vec{b}) \times \vec{c} = \alpha\, \vec{a} \times \vec{c} + \beta\, \vec{b} \times \vec{c}$

$\vec{a} \times (\alpha \vec{b} + \beta \vec{c}) = \alpha\, \vec{a} \times \vec{b} + \beta\, \vec{a} \times \vec{c}$

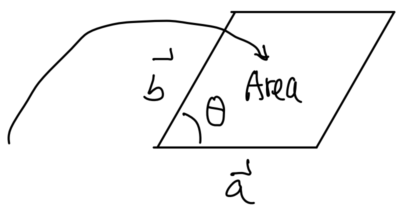
(3) $(\vec{a} \times \vec{b}) \cdot \vec{a} = (\vec{a} \times \vec{b}) \cdot \vec{b} = 0$ (easy from definition) (Ex)

Geometric

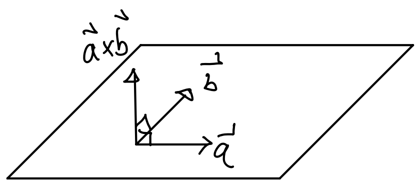
(4) Let θ = angle between $\vec{a}$ & $\vec{b}$, then

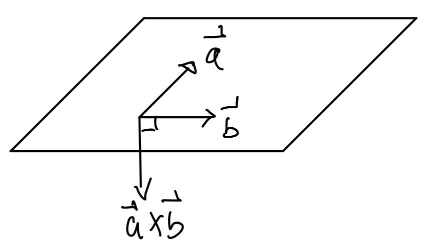
$\| \vec{a} \times \vec{b} \| = \| \vec{a} \| \| \vec{b} \| \sin \theta$

= Area of the parallelogram spanned by $\vec{a}$ & $\vec{b}$



Remarks (i) Formula (3) $\Rightarrow$ $\vec{a} \times \vec{b} \perp \vec{a}$ & $\vec{a} \times \vec{b} \perp \vec{b}$





(Also $\vec{a}, \vec{b}$, $\vec{a} \times \vec{b}$ satisfy right-hand rule by checking the defn.)

(ii) Formula (4): $\vec{a}\times\vec{b}=\vec{0} \Leftrightarrow \text{Area}(▱)=0$

$\Leftrightarrow \vec{a}=r\vec{b}$ or $\vec{b}=r\vec{a}$ for some $r\in\mathbb{R}$

$\Leftrightarrow \{\vec{a},\vec{b}\}$ is linearly dependent

(linear algebra)

Pf of (4):

By straight forward calculation (explaning both sides using defn.)

$$\|\vec{a}\times\vec{b}\|^2=\left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3\end{vmatrix}\right)^2+\left(-\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3\end{vmatrix}\right)^2+\left(\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2\end{vmatrix}\right)^2$$

$$= \cdots\cdots \quad (\text{Ex!})$$

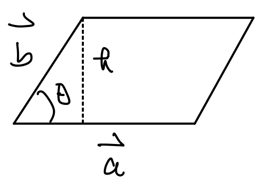
$$= (a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)-(a_1b_1+a_2b_2+a_3b_3)^2$$

$$= \|\vec{a}\|^2\|\vec{b}\|^2-(\vec{a}\cdot\vec{b})^2$$

$$= \|\vec{a}\|^2\|\vec{b}\|^2-(\|\vec{a}\|\|\vec{b}\|\cos\theta)^2$$

$$= \|\vec{a}\|^2\|\vec{b}\|^2\sin^2\theta$$

$$\Rightarrow \quad \|\vec{a}\times\vec{b}\|=\|\vec{a}\|\|\vec{b}\|\sin\theta \qquad (0\le\theta\le\pi)$$



$h=\|\vec{b}\|\sin\theta$

$\Rightarrow \text{Area}(▱)=h\cdot\|\vec{a}\|=\|\vec{a}\|\|\vec{b}\|\sin\theta$

$=\|\vec{a}\times\vec{b}\|$.

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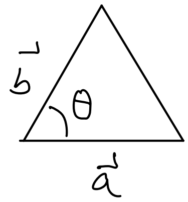
Remarks: (i) Area of 

$= \frac{1}{2}\|\vec{a}\times\vec{b}\|$

(ii) $\vec{a}, \vec{b} \in \mathbb{R}^2$, i.e. $\vec{a} = (a_1, a_2, 0)$

$\vec{b} = (b_1, b_2, 0)$

$$\vec{a}\times\vec{b} = \left(0, 0, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\right)$$

$$\Rightarrow \text{Area}(\,▱\,) = \left|\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\right|$$

absolute value of the 2x2 determinant

$$= \left|\det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\right|$$