Math 2010A Advanced Calculus I

<u>Differential</u> Calculus of Functions of Several variables.

$$\underbrace{f(x,y)\ ,\ f(x,y,z)}_{\text{mainly}}\ ,\ \underline{f(x_1,\cdots,x_n)}$$

$n = 2, 3$ (most of the times)

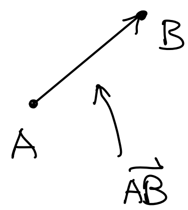
<u>Vectors in $\mathbb{R}^n$</u> : $\vec{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$

$$\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R} \quad (n \text{ copies of } \mathbb{R})$$

$$= \{(x_1, \cdots, x_n) : x_i \in \mathbb{R} \text{ for } i = 1, \cdots, n\}$$

<u>Remarks</u> : (i) In Textbook, "bold face" $\mathbf{x}$ is used to denote a vector.

(ii) In high school, $\overrightarrow{AB}$ is a vector with initial point A & terminal point B.

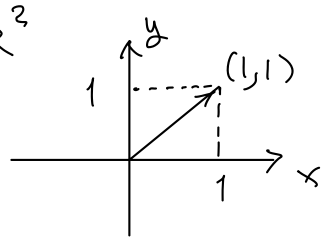



For $\vec{x} = (x_1, \cdots, x_n)$ is a vector with initial point $A = 0 = (0, \cdots, 0)$ & terminal point $B = (x_1, \cdots, x_n)$.

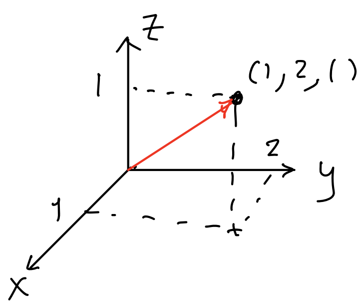
(iii) $\mathbb{R}^n$ is called <u>Cartesian product</u> of n-copies of $\mathbb{R}$.

$(x_1, \cdots, x_n)$ <u>Cartesian / rectangular coordinates</u> of the point.

eg: $\mathbb{R}^2$




$\mathbb{R}^3$



# Basic Operations of Vectors

Let $\vec{a} = (a_1, \cdots, a_n) \in \mathbb{R}^n$

$\vec{b} = (b_1, \cdots, b_n) \in \mathbb{R}^n$

$r \in \mathbb{R}$

## Equality

$$\vec{a} = \vec{b} \iff a_i = b_i \text{ for } i = 1, \cdots, n.$$

## Addition

$$\vec{a} + \vec{b} = (a_1 + b_1, \cdots, a_n + b_n)$$

## Scalar Multiplication

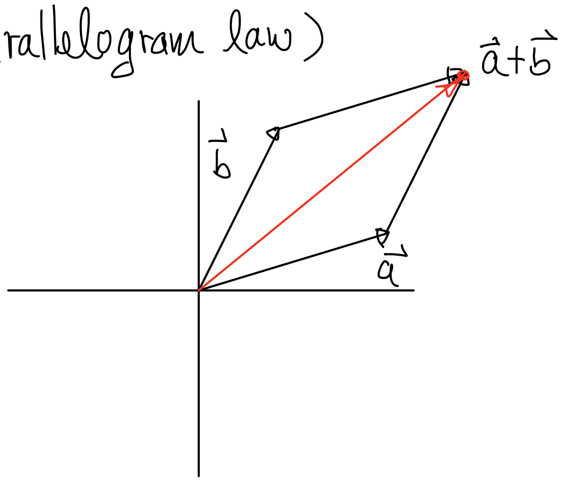
$$r\vec{a} = (ra_1, \cdots, ra_n)$$

## Subtraction

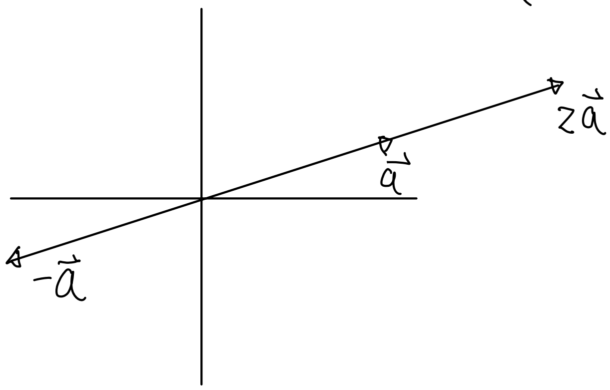
$$\vec{a} - \vec{b} = \vec{a} + (-1)\vec{b}$$

$$= (a_1 - b_1, \cdots, a_n - b_n)$$

(Parallelogram law)




(2=r)




(Ex: what is $\vec{a} - \vec{b}$?)

## Length, Dot (Inner) Product in $\mathbb{R}^n$

<u>Defn</u>: Let $\vec{a} = (a_1, \cdots, a_n) \in \mathbb{R}^n$

$\vec{b} = (b_1, \cdots, b_n) \in \mathbb{R}^n$

Then <u>dot product</u> (or <u>inner product</u>)

$$\vec{a} \cdot \vec{b} = a_1 b_1 + \cdots + a_n b_n$$

(Have to write the dot!)

In particular,

$$\vec{a} \cdot \vec{a} = a_1^2 + \cdots + a_n^2 \overset{\text{def}}{=} \|\vec{a}\|^2$$

$\|\vec{a}\| = \sqrt{a_1^2 + \cdots + a_n^2}$ is called the <u>length/magnitude</u> of the vector $\vec{a}$.

(In Textbook, it's denoted by $|\vec{a}|$.)

<u>Remark</u>: Another notation for dot (inner) product

$$\langle \vec{a}, \vec{b} \rangle = \vec{a} \cdot \vec{b}$$

## Properties of Dot Product

Let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^n$, $r \in \mathbb{R}$. Then

(1) $(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$

$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

(2) $(r\vec{a}) \cdot \vec{b} = \vec{a} \cdot (r\vec{b}) = r(\vec{a} \cdot \vec{b})$

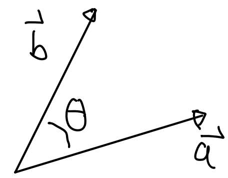
(3) $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

(4) $\vec{a}\cdot\vec{a} \geq 0$ and "equality holds" $\Leftrightarrow$ $\vec{a} = \vec{0} = (0, \cdots, 0)$ zero vector

(5) $\vec{a}\cdot\vec{a} = \|\vec{a}\|^2$

(6) $\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$

where $\theta$ = angle between $\vec{a}$ & $\vec{b}$




Hence, if $\vec{a}, \vec{b} \neq \vec{0}$,

$$\vec{a}\cdot\vec{b} = 0 \Leftrightarrow \cos\theta = 0 \Leftrightarrow \vec{a} \perp \vec{b} \text{ (perpendicular)}$$

<u>Remarks</u> (i) For $n = 2, 3$, (5) & (6) are geometric properties of $\mathbb{R}^n$.

(ii) For $n \geq 4$, $\|\vec{a}\|$ in (5) & (6) is defined before and the $\theta$ in (6) is the definition

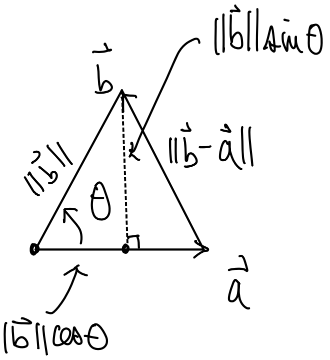
$$\theta \overset{def}{=} \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|}\right)$$ for high dimensional vectors

(iii) If $\|\vec{u}\| = 1$, then $\vec{u}$ is called a <u>unit vector</u>.

<u>eg</u>: $\hat{i} = (1, 0, 0)$, $\hat{j} = (0, 1, 0)$, $\hat{k} = (0, 0, 1)$ are unit vectors in $\mathbb{R}^3$.

<u>Pf of (6)</u> (for $n \leq 3$)

(Assume $0 < \theta \leq \frac{\pi}{2}$)



$$\|\vec{b}-\vec{a}\|^2 = (\vec{b}-\vec{a})\cdot(\vec{b}-\vec{a}) = \vec{b}\cdot\vec{b} - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{a} + \vec{a}\cdot\vec{a}$$

$$= \|\vec{b}\|^2 - 2\vec{a}\cdot\vec{b} + \|\vec{a}\|^2$$

From the figure

$$\|\vec{b}-\vec{a}\|^2 = (\|\vec{a}\| - \|\vec{b}\|\cos\theta)^2 + \|\vec{b}\|^2\sin^2\theta$$

$$= \cdots = \|\vec{a}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\theta + \|\vec{b}\|^2$$

$$\Rightarrow \quad \vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta \quad ※$$

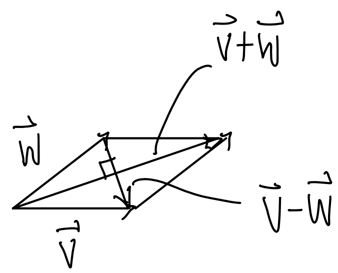
eg: Suppose $\vec{v}$ & $\vec{w}$ are vectors of the same length.
Show that $(\vec{v}+\vec{w})\cdot(\vec{v}-\vec{w}) = 0$.

Soln: $(\vec{v}+\vec{w})\cdot(\vec{v}-\vec{w}) = \vec{v}\cdot\vec{v} + \vec{w}\cdot\vec{v} - \vec{v}\cdot\vec{w} - \vec{w}\cdot\vec{w}$

$= \|\vec{v}\|^2 - \|\vec{w}\|^2 = 0$ by assumption that $\|\vec{v}\| = \|\vec{w}\|$. ※

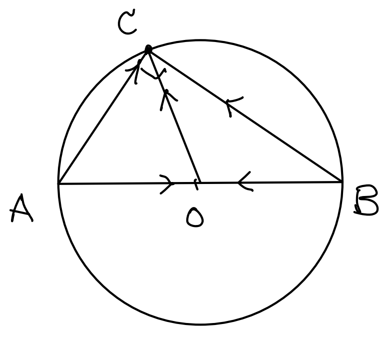
Remark:

$\|\vec{v}\| = \|\vec{w}\|$




Geometric meaning:
Diagonals of a rhombus are perpendicular.

eg:




AB is diameter, C on the circle
show that $\angle ACB = 90° = \frac{\pi}{2}$

Soln: $\vec{AC} = \vec{AO} + \vec{OC}$

$\vec{BC} = \vec{BO} + \vec{OC}$

$AB = \text{diameter} \Rightarrow \vec{AO} = -\vec{BO}$

Then $\vec{AC} \cdot \vec{BC} = (\vec{AO} + \vec{OC}) \cdot (\vec{BO} + \vec{OC})$

$= (\vec{AO} + \vec{OC}) \cdot (-\vec{AO} + \vec{OC})$

$= -\|\vec{AO}\|^2 + \|\vec{OC}\|^2$

$= 0$ because A & C are on the circle & O = center.

$\Rightarrow \angle ACB = \frac{\pi}{2}$

#

Two inequalities (Au's Book §1.2)

**Cauchy-Schwarz Inequality**

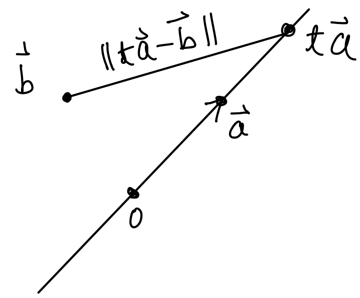
Let $\vec{a}, \vec{b} \in \mathbb{R}^n$. Then $|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\| \, \|\vec{b}\|$

i.e. $\left| \sum_{i=1}^{n} a_i b_i \right| \leq \sqrt{\sum_{i=1}^{n} a_i^2} \sqrt{\sum_{i=1}^{n} b_i^2}$

Equality holds $|\vec{a} \cdot \vec{b}| = \|\vec{a}\| \|\vec{b}\|$ $\iff$ $\vec{a} = r\vec{b}$ or $\vec{b} = r\vec{a}$ for some $r \in \mathbb{R}$

Pf: Case 1 If $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then both sides are 0, equality holds & $\vec{a}=0\vec{b}$ or $\vec{b}=0\vec{a}$.

Case 2 $\vec{a}\neq\vec{0}$ & $\vec{b}\neq\vec{0}$




Let $f(t) = \|t\vec{a}-\vec{b}\|^2 \geq 0$ for $t\in\mathbb{R}$

Then $0 \leq f(t) = (t\vec{a}-\vec{b})\cdot(t\vec{a}-\vec{b})$

$= t^2\|\vec{a}\|^2 - 2t\,\vec{a}\cdot\vec{b} + \|\vec{b}\|^2$

$\Rightarrow \quad \Delta = (-2\vec{a}\cdot\vec{b})^2 - 4\|\vec{a}\|^2\|\vec{b}\|^2 \leq 0$

(discriminant of the quadratic)

$\Rightarrow \quad |\vec{a}\cdot\vec{b}| \leq \|\vec{a}\|\|\vec{b}\|$

Equality holds $\Leftrightarrow \Delta = 0$

$\Leftrightarrow$ $f(t)$ has a repeated root $r\in\mathbb{R}$

$\Leftrightarrow \|r\vec{a}-\vec{b}\|^2 = f(r) = 0$

$\Leftrightarrow \vec{b} = r\vec{a}$. #

Remark: Cauchy-Schwarz inequality $\Rightarrow$

$$-1 \leq \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|} \leq 1 \qquad (\text{provided } \vec{a}\neq\vec{0}, \vec{b}\neq\vec{0})$$

$\Rightarrow$ The formula $\theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|}\right)$ defining the angle between $\vec{a}$ & $\vec{b}$ (in dim. $n\geq 4$) is well-defined.

(If $n\leq 3$, we've proved the formula)