MATH 2010 Advanced Calculus
Suggested Solution of Homework 7

Exercises 14.6

Q4 Solution:

$$7 = x^2 + 2xy - y^2 + z^2 =: f(x, y, z), \quad P_0 = (1, -1, 3).$$

Normal direction at P_0 is given by the gradient of f at P_0

$$\nabla f(P_0) = (\partial_x f, \partial_y f, \partial_z f)\big|_{P_0} = (2x + 2y, 2x - 2y, 2z)\big|_{P_0} = (0, 4, 6).$$

(a) Tangent Plane to $f(x, y, z) = 7$ at P_0 is

$$\partial_x f(P_0)(x - x_0) + \partial_y f(P_0)(y - y_0) + \partial_z f(P_0)(z - z_0) = 0, \quad \Rightarrow \quad 2y + 3z - 7 = 0.$$

Alternatively, since normal direction is $(0, 4, 6)$, the tangent plane must take the following form

$$0 \cdot x + 4 \cdot y + 6 \cdot z = c.$$

Since $P_0 = (1, -1, 3)$ is on this plane, one has $c = 14$, which gives the same result;

(b) Normal Line to $f(x, y, z) = 7$ at P_0 is

$$L(t) = P_0 + t\nabla f(P_0) = (1, -1 + 4t, 3 + 6t), \quad t \in \mathbb{R}.$$

□

Q6 Solution:

$$0 = x^2 - xy - y^2 - z =: f(x, y, z), \quad P_0 = (1, 1, -1).$$

Normal direction at P_0 is given by the gradient of f at P_0

$$\nabla f(P_0) = (\partial_x f, \partial_y f, \partial_z f)\big|_{P_0} = (2x - y, -x - 2y, -1)\big|_{P_0} = (1, -3, -1).$$

(a) Tangent Plane to $f(x, y, z) = 0$ at P_0 is

$$x - 3y - z + 1 = 0,$$

exact the same steps as Q4 (a);

(b) Normal Line to $f(x,y,z)=0$ at P_0 is

$$L(t)=P_0+t\nabla f(P_0)=(1+t,1-3t,-1-t),\quad t\in\mathbb{R}.$$

$\square$

Q10 Solution:

$$0=e^{-(x^2+y^2)}-z=:f(x,y,z),\quad P_0=(0,0,1).$$

Normal direction at P_0 is given by the gradient of f at P_0

$$\nabla f(P_0)=(\partial_x f,\partial_y f,\partial_z f)\big|_{P_0}=\big(-2xe^{-(x^2+y^2)},-2ye^{-(x^2+y^2)},-1\big)\big|_{P_0}=(0,0,-1).$$

Tangent Plane to $f(x,y,z)=0$ at P_0 is

$$z-1=0.$$

$\square$

Q12 Solution:

$$0=4x^2+y^2-z=:f(x,y,z),\quad P_0=(1,1,5).$$

Normal direction at P_0 is given by the gradient of f at P_0

$$\nabla f(P_0)=(\partial_x f,\partial_y f,\partial_z f)\big|_{P_0}=\big(8x,2y,-1\big)\big|_{P_0}=(8,2,-1).$$

Tangent Plane to $f(x,y,z)=0$ at P_0 is

$$8x+2y-z-5=0.$$

$\square$

Exercises 14.7

Q32 Solution:

$$D(x,y)=x^2-xy+y^2+1.$$

Compute the first order derivatives

$$\partial_x D=2x-y,\quad \partial_y D=-x+2y.$$

By solving the linear system

$$\begin{cases}0=\partial_x D=2x-y,\\ 0=\partial_y D=-x+2y,\end{cases}$$

one obtain the critical point $(0,0)$. And note that

$$\partial_x^2 D=2,\quad \partial_{xy}D=-1,\quad \partial_y^2 D=2,$$

so $\partial_x^2 D > 0$, and $\det(\partial^2 D) > 0$, one can conclude that $D(0,0) = 1$ is the local minimum. In fact this is the global minimum, since

$$D(x,y) = x^2 - xy + y^2 + 1 = x^2 - xy + \frac{1}{4}y^2 + \frac{3}{4}y^2 + 1 = \left(x - \frac{1}{2}y\right)^2 + \frac{3}{4}y^2 + 1 \geq 1.$$

For absolute maxima, it suffices to check the boundary because D can not achieve the maxima in the interior, otherwise it leads to contradiction.

- On $x = 0$, $D\big|_{x=0} = y^2 + 1$, and $\max_{x=0} D = D(0,4) = 17$;
- On $y = 4$, $D\big|_{y=4} = x^2 - 4x + 17 = (x-2)^2 + 13$, and $\max_{y=4} D = D(0,4) = D(4,4) = 17$;
- On $x = y$, $D\big|_{y=x} = x^2 + 1$, and $\max_{x=y} D = D(4,4) = 17$.

In conclusion,

(A) D achieve its absolute minima at $(0,0)$, and the absolute minimum is 1;

(B) D achieve its absolute maxima at $(0,4)$ and $(4,4)$, and the absolute maximum is 17.

□

Q34 Solution:

$$T(x,y) = x^2 + xy + y^2 - 6x.$$

Compute the first order derivatives

$$\partial_x T = 2x + y - 6, \quad \partial_y T = x + 2y.$$

By solving the linear system

$$\begin{cases} 0 = \partial_x T = 2x + y - 6, \\ 0 = \partial_y T = x + 2y, \end{cases}$$

one obtain the critical point $(4,-2)$. And note that

$$\partial_x^2 T = 2, \quad \partial_{xy} T = 1, \quad \partial_y^2 T = 2,$$

so $\partial_x^2 T > 0$, and $\det(\partial^2 T) > 0$, one can conclude that $T(4,-2) = -12$ is the local minimum. In fact this is the global minimum, since

$$T(x,y) = x^2 + xy + y^2 - 6x = \frac{1}{4}x^2 + xy + y^2 + \frac{3}{4}x^2 - 6x + 12 - 12 = \left(\frac{1}{2}x + y\right)^2 + \frac{3}{4}(x-4)^2 - 12 \geq -12.$$

For absolute maxima, it suffices to check the boundary because T can not achieve the maxima in the interior:

- On $x = 0$, $T\big|_{x=0} = y^2$, and $\max_{x=0} T = T(0,3) = T(0,-3) = 9$;
- On $y = -3$, $T\big|_{y=-3} = x^2 - 9x + 9 = \left(x - \frac{9}{2}\right)^2 - \frac{45}{4}$, and $\max_{y=-3} T = T(0,-3) = 9$;
- On $y = 3$, $T\big|_{y=3} = x^2 - 3x + 9 = \left(x - \frac{3}{2}\right)^2 + \frac{27}{4}$, and $\max_{y=3} T = T(5,3) = 19$;
- On $x = 5$, $T\big|_{x=5} = y^2 + 5y - 5 = \left(y + \frac{5}{2}\right)^2 - \frac{45}{4}$, and $\max_{x=5} T = T(5,3) = 19$;

In conclusion,

(A) T achieve its absolute minima at $(4,-2)$, and the absolute minimum is -12;

(B) T achieve its absolute maxima at $(5,3)$, and the absolute maximum is 19.

□

Q36 Solution:

$$f(x,y) = 48xy - 32x^3 - 24y^2.$$

Compute the first order derivatives

$$\partial_x f = 48y - 96x^2, \quad \partial_y f = 48x - 48y.$$

By solving the system

$$\begin{cases} 0 = \partial_x f = 48y - 96x^2, \\ 0 = \partial_y f = 48x - 48y, \end{cases}$$

one obtain the critical points $(0,0)$ and $\left(\frac{1}{2},\frac{1}{2}\right)$. And note that

$$\partial_x^2 f = -192x, \quad \partial_{xy} f = 48, \quad \partial_y^2 f = -48,$$

so $\det(\partial^2 f) = 48^2(4x-1)$, and

- $\det(\partial^2 f)(0,0) < 0$, f has a saddle point at $(0,0)$;
- $\det(\partial^2 f)\left(\frac{1}{2},\frac{1}{2}\right) > 0$ and $\partial_x^2 f\left(\frac{1}{2},\frac{1}{2}\right)<0$, so f has a local maximum at $\left(\frac{1}{2},\frac{1}{2}\right)$, and $f\left(\frac{1}{2},\frac{1}{2}\right) = 2$.

About the boundary:

- On $x=0$, $f\big|_{x=0} = -24y^2$, and $\max_{x=0} f = f(0,0) = 0$, $\min_{x=0} f = f(0,1) = -24$;
- On $y=0$, $f\big|_{y=0} = -32x^3$, and $\max_{y=0} f = f(0,0) = 0$, $\min_{y=0} f = f(1,0) = -32$;
- On $y=1$, $f\big|_{y=1} = 48x - 32x^3 - 24$, note that

$$\frac{\mathrm{d}}{\mathrm{d}x} f\Big|_{y=1} = 48 - 96x^2, \quad \frac{\mathrm{d}}{\mathrm{d}x} f\Big|_{y=1}\left(x = \frac{\sqrt{2}}{2}\right) = 0,$$

  so $\max_{y=1} f = f\left(\frac{\sqrt{2}}{2},1\right) = 16\sqrt{2} - 24$, $\min_{y=1} f = f(0,1) = -24$;
- On $x=1$, $f\big|_{x=1} = 48y - 32 - 24y^2$, note that

$$48y - 32 - 24y^2 = -24(y-1)^2 - 8,$$

  so $\max_{x=1} f = f(1,1) = -8$, $\min_{x=1} f = f(1,0) = -32$;

In conclusion,

(A) f achieve its absolute maxima at $\left(\frac{1}{2},\frac{1}{2}\right)$, and the absolute maxima is 2;

(B) f achieve its absolute minima at $(1,0)$, and the absolute minima is -32.

□

Q39 Solution: In order for the integral to achieve its largest value, it suffices to ensure the non-negativity of the integrand. Since

$$6 - x - x^2 \geq 0, \quad \text{for } -3 \leq x \leq 2, \text{ and } < 0 \text{ otherwise},$$

so the integral achieves its largest value for $a = -3$, $b = 2$.

Alternatively one can treat the integral as a multivariable function

$$f(a, b) := \int_a^b \left(6 - x - x^2\right) \mathrm{d}x$$

therefore

$$\partial_a f = -6 + a + a^2, \quad \partial_b f = 6 - b - b^2.$$

By solving the system

$$\begin{cases} 0 = \partial_a f = -6 + a + a^2, \\ 0 = \partial_b f = 6 - b - b^2, \end{cases}$$

one obtain the critical points $(-3, -3)$, $(-3, 2)$, $(2, -3)$, and $(2, 2)$. The region we are considering is $a \leq b$, therefore $(2, -3)$ is omitted. Note that

$$\partial_a^2 f = 1 + 2a, \quad \partial_{ab} f = 0, \quad \partial_b^2 f = -1 - 2b,$$

so $\det(\partial^2 f) = -(1 + 2a)(1 + 2b)$, and

- $\det(\partial^2 f)(-3, -3) < 0$, $\det(\partial^2 f)(2, 2) < 0$, f has saddle points at $(-3, -3)$ and $(2, 2)$;
- $\det(\partial^2 f)(-3, 2) = 25 > 0$ and $\partial_x^2 f(-3, 2) = -5 < 0$, so f has a local maximum at $(-3, 2)$, and $f(-3, 2) = \frac{125}{6} = 20\frac{5}{6}$.

Note that on the boundary $a = b$, $f \equiv 0 < f(-3, 2)$, so the integral has its largest value at $a = -3$, $b = 2$. □

Q42 Solution:

$$f(x, y) = xy + 2x - \ln(x^2 y).$$

On the first quadrant

$$f(x, y) = xy + 2x - 2\ln(x) - \ln(y).$$

Compute the first order derivatives

$$\partial_x f = y + 2 - \frac{2}{x}, \quad \partial_y f = x - \frac{1}{y}.$$

By solving the system

$$\begin{cases} 0 = \partial_x f = y + 2 - \dfrac{2}{x}, \\ 0 = \partial_y f = x - \dfrac{1}{y}, \end{cases}$$

one obtain the critical point $\left(\frac{1}{2}, 2\right)$. Note that

$$\partial_x^2 f = \frac{2}{x^2}, \quad \partial_{xy} f = 1, \quad \partial_y^2 f = \frac{1}{y^2},$$

so $\det(\partial^2 f) = \frac{2}{x^2y^2} - 1$, and

$$\det(\partial^2 f)\Big(\frac{1}{2}, 2\Big) = 1 > 0, \quad \partial_x^2 f\Big(\frac{1}{2}, 2\Big) = 8 > 0.$$

Consequently f takes on a local minimum at $\left(\frac{1}{2}, 2\right)$.

In fact f takes the global minimum at $\left(\frac{1}{2}, 2\right)$ in the first quadrant: Do the change of variables

$$\begin{cases} a = x & 0 < a < +\infty, \\ b = xy & 0 < b < +\infty, \end{cases}$$

then

$$f = f(a, b) = b + 2a - \ln(a) - \ln(b).$$

This change of variables separates the two variables, and note that

$$b - \ln(b) \geq 1, \quad 2a - \ln(a) \geq 1 + \ln(2),$$

so

$$f \geq 2 + \ln(2) = f\big(x = 1/2, y = 2\big).$$

□

Q62 Solution: Use the parametric equations $x = 3\cos(t)$, $y = 2\sin(t)$.

(i) The semiellipse $0 \leq t \leq \pi$;

  (a) $f(t) = 6\cos(t) + 6\sin(t) = 6\sqrt{2}\sin(t + \pi/4)$, so $\max f = f(\pi/4) = 6\sqrt{2}$, and $\min f = f(\pi) = -6$;

  (b) $g(t) = 6\cos(t)\sin(t) = 3\sin(2t)$, so $\max g = g(\pi/4) = 3$, and $\min g = g(3\pi/4) = -3$;

  (c) $h(t) = 9\cos^2(t) + 12\sin^2(t) = -\frac{3}{2}\cos(2t) + \frac{21}{2}$ $\big($or simply $h(t) = 9 + 3\sin^2(t)\big)$, so $\max h = h(\pi/2) = 12$, and $\min h = h(0) = h(\pi) = 9$;

(ii) The quarter ellipse $0 \leq t \leq \pi/2$;

  (a) Again $\max f = f(\pi/4) = 6\sqrt{2}$, but $\min f = f(0) = f(\pi/2) = 6$;

  (b) Again $\max g = g(\pi/4) = 3$, but $\min g = g(0) = g(\pi/2) = 0$;

  (c) Again $\max h = h(\pi/2) = 12$, and $\min h = h(0) = 9$.

□