MATH 2010 Advanced Calculus
Suggested Solution of Homework 6

Exercises 14.5

Q5 Solution:

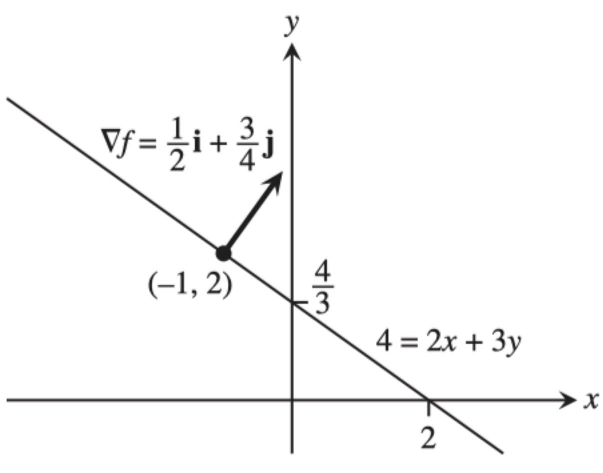
$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(\frac{1}{\sqrt{2x+3y}}, \frac{3/2}{\sqrt{2x+3y}}\right).$$

Evaluation

$$\nabla f(-1,2) = \left(\frac{1}{2}, \frac{3}{4}\right).$$

Level curve passing through the given point is

$$f(x,y) = f(-1,2) = 2, \quad \Rightarrow \quad 2x+3y = 4.$$



□

Q8 Solution:

$$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(-6zx + \frac{1}{1+x^2z^2} \cdot z\ ,\ -6zy\ ,\ 6z^2 - 3(x^2+y^2) + \frac{1}{1+x^2z^2} \cdot x\right)$$

Evaluation

$$\nabla f(1,1,1) = \left(-\frac{11}{2}\ ,\ -6\ ,\ \frac{1}{2}\right).$$

□

Q15 Solution:

$$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (y+z, x+z, y+x)$$

Evaluation

$$\nabla f(1,-1,2) = (1,3,0).$$

The directional derivative of f at P_0 in $\boldsymbol{u}$ direction is given by

$$\frac{\boldsymbol{u}}{|\boldsymbol{u}|} \cdot \nabla f(P_0) = \frac{(3,6,-2)}{7} \cdot (1,3,0) = 3.$$

□

Q20 Solution:

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \big(2yx + \sin(y) y e^{xy}, x^2 + xe^{xy}\sin(y) + e^{xy}\cos(y)\big).$$

Evaluation

$$\nabla f(1,0) = (0,2).$$

The direction in which the function increases the most rapidly at P_0 is

$$\boldsymbol{u} := \frac{\nabla f(P_0)}{|\nabla f(P_0)|} = (0,1),$$

and the derivative in $\boldsymbol{u}$ direction is

$$\boldsymbol{u} \cdot \nabla f(P_0) = \frac{\nabla f(P_0)}{|\nabla f(P_0)|} \cdot \nabla f(P_0) = |\nabla f(P_0)| = 2.$$

Consequently the direction in which the function decreases the most rapidly at P_0 is $-\boldsymbol{u} = -(0,1)$, and the derivative in $-\boldsymbol{u}$ direction is -2. □

Q30 Solution:

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(\frac{2y}{(x+y)^2}, \frac{-2x}{(x+y)^2}\right)$$

Evaluation

$$\nabla f\left(-\frac{1}{2}, \frac{3}{2}\right) = (3,1).$$

(a) By Cauchy-Schwartz inequality,

$$\big|D_{\boldsymbol{u}} f(-1/2, 3/2)\big|^2 \le \big|\boldsymbol{u}\big|^2 \big|\nabla f\,(-1/2,3/2)\,\big|^2 = 10,$$

and equality holds for $\boldsymbol{u} // \nabla f\,(-1/2,3/2)$, i.e. $\max D_{\boldsymbol{u}} f(-1/2,3/2) = \sqrt{10}$, with $\boldsymbol{u} = \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)$;

(b) Conversely, $\min D_{\boldsymbol{u}} f(-1/2,3/2) = -\sqrt{10}$, with $\boldsymbol{u} = \left(-\frac{3}{\sqrt{10}}, -\frac{1}{\sqrt{10}}\right)$;

(c) $D_{\boldsymbol{u}} f(-1/2,3/2) = 0$ implies $\boldsymbol{u}$ is orthogonal to $\nabla f\,(-1/2,3/2)$, i.e. $\boldsymbol{u} = \left(-\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$ or $\left(\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}\right)$;

(d) $\boldsymbol{u} \cdot (3,1) = -2$, together with $\big|\boldsymbol{u}\big| = 1$ implies $\boldsymbol{u} = \left(\frac{-6+\sqrt{6}}{10}, \frac{-3\sqrt{6}-2}{10}\right)$ or $\left(\frac{-6-\sqrt{6}}{10}, \frac{3\sqrt{6}-2}{10}\right)$;

(e) $\boldsymbol{u} \cdot (3,1) = 1$, together with $\big|\boldsymbol{u}\big| = 1$ implies $\boldsymbol{u} = (0,1)$ or $(3/5, -4/5)$.

□

Exercises 14.6

Q20 Solution:

$$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \big(e^x\cos(yz), -ze^x\sin(yz), -ye^x\sin(yz)\big).$$

Evaluation at the origin

$$\nabla f(0,0,0) = (1,0,0)\,.$$

The change is given by

$$\frac{\boldsymbol{u}}{|\boldsymbol{u}|}\cdot\nabla f(0,0,0)\,\mathrm{d}s = \frac{(2,2,-2)}{2\sqrt{3}}\cdot(1,0,0)\cdot 0.1 = \frac{1}{10\sqrt{3}} \approx 0.0577.$$

□

Q28 Solution:

$$\partial_x f = 3y^4x^2, \quad \partial_y f = 4x^3y^3.$$

The linearization is given by

$$L_{P_0}(x,y) = f(P_0) + \partial_x f(P_0)(x-x_0) + \partial_y f(P_0)(y-y_0).$$

So

(a) $L_{(1,1)}(x,y) = 1 + 3(x-1) + 4(y-1) = 3x + 4y - 6$;

(b) $L_{(0,0)}(x,y) = 0$.

□

Q35 Solution:

$$\partial_x f = \cos(y), \quad \partial_y f = 1 - x\sin(y).$$

The linearization at $(0,0)$ is given by

$$L_{(0,0)}(x,y) = f(0,0) + \partial_x f(0,0)(x-0) + \partial_y f(0,0)(y-0) = 1 + x + y.$$

Find the upper bound for the second order derivatives by using $\big|\cos(y)\big| \le 1$ and $\big|\sin(y)\big| \le 1$

$$\sup_R \Big\{\big|\partial_x^2 f\big|, \big|\partial_y^2 f\big|, \big|\partial_{xy} f\big|\Big\} = \sup_R \Big\{0, \big|-x\cos(y)\big|, \big|-\sin(y)\big|\Big\} \le \max\Big\{0, 0.2, 1\Big\} = 1 =: M.$$

The upper bound for the error is given by

$$\big|E(x,y)\big| \le \frac{1}{2}M\sup_R\big(\,|x-0| + |y-0|\,\big)^2 = 0.08.$$

□

Q43 Solution:

$$\partial_x f = e^x, \quad \partial_y f = -\sin(y+z), \quad \partial_z f = -\sin(y+z).$$

Linearization at P_0 of f is given by

$$L_{P_0}(x,y,z) = f(P_0) + \partial_x f(P_0)(x-x_0) + \partial_y f(P_0)(y-y_0) + \partial_z f(P_0)(z-z_0).$$

So

(a) $L_{(0,0,0)}(x,y,z) = 2 + 1(x-0) + 0(y-0) + 0(z-0) = x+2$;

(b) $L_{(0,\pi/2,0)}(x,y,z) = 1 + 1(x-0) - 1(y-\pi/2) - 1(z-0) = x-y-z+\pi/2+1$;

(c) $L_{(0,\pi/4,\pi/4)}(x,y,z) = 1 + 1(x-0) - 1(y-\pi/4) - 1(z-\pi/4) = x-y-z+\pi/2+1$.

□

Q47 Solution:

$$\partial_x f = y-3z, \quad \partial_y f = x+2z, \quad \partial_z f = 2y-3x.$$

The linearization at $(1,1,0)$ is given by

$$L_{(1,1,0)}(x,y) = f(1,1,0) + \partial_x f(1,1,0)(x-1) + \partial_y f(1,1,0)(y-1) + \partial_z f(1,1,0)(z-0) = x+y-z-1.$$

The upper bound for the second order derivatives can be derived as follows

$$M := \sup_R \left\{ \left|\partial_x^2 f\right|, \left|\partial_y^2 f\right|, \left|\partial_z^2 f\right|, \left|\partial_{xy} f\right|, \left|\partial_{yz} f\right|, \left|\partial_{zx} f\right| \right\} = 3.$$

The upper bound for the error is given by

$$\left|E\right| \le \frac{1}{2} M \sup_R \big(|x-1| + |y-1| + |z-0| \big)^2 = 0.00135.$$

□