MATH 2010 Advanced Calculus
Suggested Solution of Homework 10

Exercises 14.8

***Q26 Solution*:** Set

$$f(x,y,z) := xyz, \quad g(x,y,z) := x + y + z^2 - 16.$$

Note that

$$\nabla g(x,y,z) = (1,1,2z) \neq 0.$$

In order to maximize f under the constrain $g = 0$, compute

$$\nabla f(x,y,z) = (yz, zx, xy),$$

and find the values of x, y, z, and λ such that

$$\nabla f = \lambda \nabla g, \quad \text{and} \quad g = 0,$$

i.e. solve the following system

$$\begin{cases} yz = \lambda, & (1a) \\ zx = \lambda, & (1b) \\ xy = 2\lambda z, & (1c) \\ x + y + z^2 - 16 = 0. & (1d) \end{cases}$$

Take the product of (1a) and (1b)

$$xyz^2 = \lambda^2,$$

and compare with (1c): one can multiply both sides of (1c) with z^2

$$xyz^2 = (2\lambda z)z^2 = 2\lambda z^3,$$

and get rid of x, y

$$\lambda^2 = 2\lambda z^3. \tag{2}$$

Note that λ is on both sides of (2), one can dichotomize regarding λ:

(1) $\lambda = 0$, then

$$0 = xy = yz = zx,$$

so no matter what values x, y, z take, $f = xyz = 0$ if $\lambda = 0$;

(2) $\lambda \neq 0$, then (2) implies $\lambda = 2z^3$, i.e.

$$z = 2^{-\frac{1}{3}}\lambda^{\frac{1}{3}}. \tag{3}$$

Back to (1a) and (1b),

$$x = y = 2^{\frac{1}{3}}\lambda^{\frac{2}{3}}. \tag{4}$$

Substitute (3) and (4) into (1d),

$$2^{\frac{1}{3}}\lambda^{\frac{2}{3}} + 2^{\frac{1}{3}}\lambda^{\frac{2}{3}} + 2^{-\frac{2}{3}}\lambda^{\frac{2}{3}} - 16 = 0,$$

i.e.

$$\lambda^{\frac{2}{3}} = \frac{16}{5}2^{\frac{2}{3}}, \quad \Longrightarrow \quad \lambda^{\frac{1}{3}} = \pm\frac{4}{\sqrt{5}}2^{\frac{1}{3}},$$

consequently

$$f(x,y,z) = xyz = 2\lambda z^2 = 2\lambda 2^{-\frac{2}{3}}\lambda^{\frac{2}{3}} = 2^{\frac{1}{3}}\lambda^{\frac{5}{3}} = 2^{\frac{1}{3}}\Big(\pm\frac{4}{\sqrt{5}}2^{\frac{1}{3}}\Big)^5 = \pm\frac{2^{12}}{5^2\sqrt{5}} = \pm\frac{2^{12}}{5^3}\sqrt{5}.$$

Combine the two cases, the maxima of f under the constrain $g = 0$ is

$$\frac{2^{12}}{5^3}\sqrt{5} = \frac{4096}{125}\sqrt{5}.$$

□

Q29 Solution: Set

$$g(x,y,z) := 4x^2 + y^2 + 4z^2 - 16.$$

Note that

$$\nabla g(x,y,z) = (8x, 2y, 8z) \neq 0 \quad \text{when } g(x,y,z) = 0.$$

In order to maximize

$$T(x,y,z) = 8x^2 + 4yz - 16z + 600$$

under the constrain $g = 0$, compute

$$\nabla T(x,y,z) = (16x, 4z, 4y - 16),$$

and find the values of x, y, z, and λ such that

$$\nabla T = \lambda \nabla g, \quad \text{and} \quad g = 0,$$

i.e. solve the following system

$$\begin{cases} 16x = 8\lambda x, & \text{(5a)} \\ 4z = 2\lambda y, & \text{(5b)} \\ 4y - 16 = 8\lambda z, & \text{(5c)} \\ 4x^2 + y^2 + 4z^2 - 16 = 0. & \text{(5d)} \end{cases}$$

Note that x is on both sides of (5a), one can dichotomize regarding x:

(1) if $x = 0$, then system (5) is reduced to

$$\begin{cases} 2z = \lambda y, & \text{(6a)} \\ y - 4 = 2\lambda z, & \text{(6b)} \\ y^2 + 4z^2 - 16 = 0. & \text{(6c)} \end{cases}$$

Solve (6a) and (6b) by treating λ as a parameter

$$y(1 - \lambda^2) = 4, \tag{7}$$

$$z(1 - \lambda^2) = 2\lambda, \tag{8}$$

and then insert (7) and (8) back to (6c) to solve for λ

$$\frac{4^2}{(1-\lambda^2)^2} + 4 \cdot \frac{4\lambda^2}{(1-\lambda^2)^2} = 16, \quad \implies \quad (1-\lambda^2)^2 - \lambda^2 - 1 = 0, \quad \text{i.e. } \lambda^2(\lambda^2 - 3) = 0,$$

so in view of the values of λ, there are three subcases:

(i) $\lambda = 0$, consequently $y = 4$, $z = 0$, and

$$T(0, 4, 0) = 600;$$

(ii) $\lambda = \sqrt{3}$, consequently $y = -2$, $z = -\sqrt{3}$, and

$$T(0, -2, -\sqrt{3}) = 600 + 24\sqrt{3};$$

(iii) $\lambda = -\sqrt{3}$, consequently $y = -2$, $z = \sqrt{3}$, and

$$T(0, -2, \sqrt{3}) = 600 - 24\sqrt{3};$$

(2) If $x \neq 0$, then (5a) implies $\lambda = 2$, and one can solve (5b) and (5c) for y and z, since the subsystem (5b)-(5c) is independent of x:

$$y = z = -\frac{4}{3}.$$

Back to the constrain (5d), one can solve for x

$$x = \pm\frac{4}{3}.$$

Evaluate,

$$T\Big(\pm\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\Big) = 642\frac{2}{3}.$$

Since $24\sqrt{3} < 42\frac{2}{3}$, the hottest points are $\Big(\pm\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\Big)$. □

Q30 Solution: Set

$$g(x, y, z) := x^2 + y^2 + z^2 - 1.$$

Note that

$$\nabla g(x, y, z) = (2x, 2y, 2z) \neq 0 \quad \text{for } g(x, y, z) = 0.$$

In order to find the maxima and minima of

$$T(x,y,z) = 400xyz^2$$

under the constrain $g = 0$, compute

$$\nabla T(x,y,z) = (400yz^2, 400xz^2, 800xyz),$$

and find the values of x, y, z, and λ such that

$$\nabla T = \lambda \nabla g, \quad \text{and} \quad g = 0,$$

i.e. solve the following system

$$\begin{cases} 400yz^2 = 2\lambda x, & \text{(9a)} \\ 400xz^2 = 2\lambda y, & \text{(9b)} \\ 800xyz = 2\lambda z, & \text{(9c)} \\ x^2 + y^2 + z^2 - 1 = 0. & \text{(9d)} \end{cases}$$

Note that z is on both sides of (9c), one can dichotomize regarding z:

(1) if $z = 0$, then system (9) is reduced to

$$\begin{cases} 0 = \lambda x, & \text{(10a)} \\ 0 = \lambda y, & \text{(10b)} \\ x^2 + y^2 - 1 = 0. & \text{(10c)} \end{cases}$$

One can see that λ must be zero, otherwise (10a)-(10b) imply $x = y = 0$ which contradicts (10c). Anyway, since $z = 0$, $T \equiv 0$ in this case;

(2) If $z \neq 0$, then (9c) implies

$$400xy = \lambda,$$

and system (9) is transformed into

$$\begin{cases} yz^2 = 2x^2y, & \text{(11a)} \\ xz^2 = 2xy^2, & \text{(11b)} \\ 400xy = \lambda, & \text{(11c)} \\ x^2 + y^2 + z^2 - 1 = 0. & \text{(11d)} \end{cases}$$

Note that y is on both sides of (11a) and x is on both sides of (11b), one can dichotomize regarding x, y:

(i) At least one of x or y is zero, then (11c) implies $\lambda = 0$, and by (11a)-(11b), $x = y = 0$, consequently by (11d) $z = 1$. Anyway $T = 0$;

(ii) $x \neq 0$ and $y \neq 0$, then system (11) is reduced to

$$\begin{cases} z^2 = 2x^2, & (12a) \\ z^2 = 2y^2, & (12b) \end{cases}$$

and one can insert (12a) and (12b) into (11d) to solve for z

$$\frac{1}{2}z^2 + \frac{1}{2}z^2 + z^2 - 1 = 0, \quad \Longrightarrow \quad z^2 = \frac{1}{2};$$

then go back to system (12) to solve for x, y:

$$xy = \begin{cases} -\frac{1}{4} & (x, y) = (-1/2, 1/2) \text{ or } (1/2, -1/2), \\ \frac{1}{4} & (x, y) = (1/2, 1/2) \text{ or } (-1/2, -1/2). \end{cases}$$

In conclusion, the highest and lowest temperatures are achieved in case (2)(ii), and the highest temperature is

$$T_{\max} = 50, \quad \text{at} \quad (x, y) = (1/2, 1/2) \text{ or } (-1/2, -1/2), \quad z = \pm\frac{\sqrt{2}}{2},$$

and the lowest temperature is

$$T_{\min} = -50, \quad \text{at} \quad (x, y) = (-1/2, 1/2) \text{ or } (1/2, -1/2), \quad z = \pm\frac{\sqrt{2}}{2}.$$

□

***Q38 Solution*:** Set

$$g_1(x, y, z) := x + 2y + 3z - 6, \quad g_2(x, y, z) := x + 3y + 9z - 9.$$

Note that

$$\nabla g_1(x, y, z) = (1, 2, 3), \quad \nabla g_2(x, y, z) = (1, 3, 9),$$

so ∇g_1 and ∇g_2 are not parallel. In order to minimize

$$f(x, y, z) = x^2 + y^2 + z^2$$

under the two constrains $g_1 = 0$ and $g_2 = 0$, compute

$$\nabla f(x, y, z) = (2x, 2y, 2z),$$

and find the values of x, y, z, and λ_1, λ_2 such that

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \quad \text{and} \quad g_1 = g_2 = 0,$$

i.e. solve the following system

$$\begin{cases} 2x = \lambda_1 + \lambda_2, & (13a) \\ 2y = 2\lambda_1 + 3\lambda_2, & (13b) \\ 2z = 3\lambda_1 + 9\lambda_2, & (13c) \\ x + 2y + 3z - 6 = 0, & (13d) \\ x + 3y + 9z - 9 = 0. & (13e) \end{cases}$$

This is a linear system of five equations with five unknowns $x, y, z, \lambda_{1,2}$, and one can solve it directly: since only the values of x, y, z are needed, one can first solve (13a)-(13b) for $\lambda_{1,2}$ in terms of x, y

$$\lambda_1 = 6x - 2y, \quad \lambda_2 = 2y - 4x,$$

then substitute $\lambda_{1,2}$ into (13c) to get rid of λ

$$9x - 6y + z = 0. \tag{14}$$

Now system (13d)-(13e)-(14) is only about x, y, z, and one can solve it to obtain

$$z = \frac{9}{59}, \quad y = \frac{123}{59}, \quad x = \frac{81}{59}.$$

So the minimum is

$$f_{\min} = f\Big(\frac{81}{59}, \frac{123}{59}, \frac{9}{59}\Big) = \frac{21771}{3481} = \frac{369}{59} = 6\frac{885}{3481} = 6\frac{15}{59}.$$

□

Q39 Solution: Set

$$g_1(x, y, z) := y + 2z - 12, \quad g_2(x, y, z) := x + y - 6, \quad \text{and} \quad f(x, y, z) = x^2 + y^2 + z^2.$$

Note that

$$\nabla g_1(x, y, z) = (0, 1, 2), \quad \nabla g_2(x, y, z) = (1, 1, 0),$$

so ∇g_1 and ∇g_2 are not parallel. In order to minimize f under the two constrains $g_1 = 0$ and $g_2 = 0$, compute

$$\nabla f(x, y, z) = (2x, 2y, 2z),$$

and find the values of x, y, z, and λ_1, λ_2 such that

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \quad \text{and} \quad g_1 = g_2 = 0,$$

i.e. solve the following system

$$\begin{cases} 2x = \lambda_2, & (15a) \\ 2y = \lambda_1 + \lambda_2, & (15b) \\ 2z = 2\lambda_1, & (15c) \\ y + 2z - 12 = 0, & (15d) \\ x + y - 6 = 0. & (15e) \end{cases}$$

This is a linear system of five equations with five unknowns $x, y, z, \lambda_{1,2}$, and one can solve it directly in the same process as Question 38 above to obtain

$$z = 4, \quad y = 4, \quad x = 2,$$

so the minimum distance to the origin is achieved at (2,4,4). □

Section 14.8, Q 40

We find the extreme values of $f(x,y,z) = x^2 + 2y - z^2$ subject to the constraints $g_1(x,y,z) = 2x - y = 0$, and $g_2(x,y,z) = y + z = 0$. Then

$$\nabla f = 2x\vec{i} + 2\vec{j} - 2z\vec{k}, \quad \nabla g_1 = 2\vec{i} - \vec{j}, \quad \nabla g_2 = \vec{j} + \vec{k}$$

The gradient equation $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$ gives

$$2x = 2\lambda, \quad 2 = -\lambda + \mu, \quad -2z = \mu.$$

Then we obtain $x = \lambda, z = -1 - \frac{x}{2}$. Substituting $y = 2x$ and $z = -1 - \frac{x}{2}$ into $g_2(x,y,z)$ yields $x = \frac{2}{3}$. Moreover, substituting $x = \frac{2}{3}$ into $g_1(x,y,z)$ yields $y = \frac{4}{3}$, which gives $z = -\frac{4}{3}$. The maximum value is $f(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}) = \frac{4}{3}$.

Section 14.8, Q 41

Let $g_1(x,y,z) = z - 1 = 0$ and $g_2(x,y,z) = x^2 + y^2 + z^2 - 10 = 0 \Rightarrow \nabla g_1 = \mathbf{k}, \nabla g_2 = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, and $\nabla f = 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k} = \lambda(\mathbf{k}) + \mu(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k})$ $\Rightarrow 2xyz = 2x\mu, x^2z = 2y\mu$, and $x^2y = 2z\mu + \lambda \Rightarrow xyz = x\mu \Rightarrow x = 0$ or $yz = \mu \Rightarrow \mu = y$ since $z = 1$.

CASE 1: $x = 0$ and $z = 1 \Rightarrow y^2 - 9 = 0$ (from g_2) $\Rightarrow y = \pm 3$ yielding the points $(0, \pm 3, 1)$.

CASE 2: $\mu = y \Rightarrow x^2z = 2y^2 \Rightarrow x^2 = 2y^2$ (since $z = 1$) $\Rightarrow 2y^2 + y^2 + 1 - 10 = 0$ (from g_2) $\Rightarrow 3y^2 - 9 = 0$

$\Rightarrow y = \pm\sqrt{3} \Rightarrow x^2 = 2\left(\pm\sqrt{3}\right)^2 \Rightarrow x = \pm\sqrt{6}$ yielding the points $\left(\pm\sqrt{6}, \pm\sqrt{3}, 1\right)$.

Now $f(0, \pm 3, 1) = 1$ and $f\left(\pm\sqrt{6}, \pm\sqrt{3}, 1\right) = 6\left(\pm\sqrt{3}\right) + 1 = 1 \pm 6\sqrt{3}$. Therefore the maximum of f is $1 + 6\sqrt{3}$ at $\left(\pm\sqrt{6}, \sqrt{3}, 1\right)$, and the minimum of f is $1 - 6\sqrt{3}$ at $\left(\pm\sqrt{6}, -\sqrt{3}, 1\right)$.

Section 14.8, Q 42

(a) Let $g_1(x,y,z) = x + y + z - 40 = 0$ and $g_2(x,y,z) = x + y - z = 0 \Rightarrow \nabla g_1 = \mathbf{i} + \mathbf{j} + \mathbf{k}, \nabla g_2 = \mathbf{i} + \mathbf{j} - \mathbf{k}$, and $\nabla w = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ so that $\nabla w = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mu(\mathbf{i} + \mathbf{j} - \mathbf{k})$ $\Rightarrow yz = \lambda + \mu, xz = \lambda + \mu$, and $xy = \lambda - \mu \Rightarrow yz = xz \Rightarrow z = 0$ or $y = x$.

CASE 1: $z = 0 \Rightarrow x + y = 40$ and $x + y = 0 \Rightarrow$ no solution.

CASE 2: $x = y \Rightarrow 2x + z - 40 = 0$ and $2x - z = 0 \Rightarrow z = 20 \Rightarrow x = 10$ and $y = 10 \Rightarrow w = (10)(10)(20)$ $= 2000$

(b) $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -2\mathbf{i} + 2\mathbf{j}$ is parallel to the line of intersection $\Rightarrow$ the line is $x = -2t + 10, y = 2t + 10,$ $z = 20$. Since $z = 20$, we see that $w = xyz = (-2t + 10)(2t + 10)(20) = \left(-4t^2 + 100\right)(20)$ which has its maximum when $t = 0 \Rightarrow x = 10, y = 10$, and $z = 20$.

Section 14.8, Q 43

Let $g_1(x, y, z) = y - x = 0$ and $g_2(x, y, z) = x^2 + y^2 + z^2 - 4 = 0$. Then $\nabla f = y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}$, $\nabla g_1 = -\mathbf{i} + \mathbf{j}$, and $\nabla g_2 = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k} = \lambda(-\mathbf{i} + \mathbf{j}) + \mu(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k})$ $\Rightarrow y = -\lambda + 2x\mu$, $x = \lambda + 2y\mu$, and $2z = 2z\mu \Rightarrow z = 0$ or $\mu = 1$.

CASE 1: $z = 0 \Rightarrow x^2 + y^2 - 4 = 0 \Rightarrow 2x^2 - 4 = 0$ (since $x = y$) $\Rightarrow x = \pm\sqrt{2}$ and $y = \pm\sqrt{2}$ yielding the points $\left(\pm\sqrt{2}, \pm\sqrt{2}, 0\right)$.

CASE 2: $\mu = 1 \Rightarrow y = -\lambda + 2x$ and $x = \lambda + 2y \Rightarrow x + y = 2(x + y) \Rightarrow 2x = 2(2x)$ since $x = y \Rightarrow x = 0 \Rightarrow y = 0$ $\Rightarrow z^2 - 4 = 0 \Rightarrow z = \pm 2$ yielding the points $(0, 0, \pm 2)$.

Now, $f(0, 0, \pm 2) = 4$ and $f\left(\pm\sqrt{2}, \pm\sqrt{2}, 0\right) = 2$. Therefore the maximum value of *f* is 4 at $(0, 0, \pm 2)$ and the minimum value of *f* is 2 at $f\left(\pm\sqrt{2}, \pm\sqrt{2}, 0\right)$.

Section 14.8, Q 47

(a) Maximize $f(a, b, c) = a^2b^2c^2$ subject to $a^2 + b^2 + c^2 = r^2$. Thus $\nabla f = 2ab^2c^2\mathbf{i} + 2a^2bc^2\mathbf{j} + 2a^2b^2c\mathbf{k}$ and $\nabla g = 2a\mathbf{i} + 2b\mathbf{j} + 2c\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2ab^2c^2 = 2a\lambda$, $2a^2bc^2 = 2b\lambda$, and $2a^2b^2c = 2c\lambda$ $\Rightarrow 2a^2b^2c^2 = 2a^2\lambda = 2b^2\lambda = 2c^2\lambda \Rightarrow \lambda = 0$ or $a^2 = b^2 = c^2$.

CASE 1: $\lambda = 0 \Rightarrow a^2b^2c^2 = 0$.

CASE 2: $a^2 = b^2 = c^2 \Rightarrow f(a, b, c) = a^2a^2a^2$ and $3a^2 = r^2 \Rightarrow f(a, b, c) = \left(\frac{r^2}{3}\right)^3$ is the maximum value.

(b) The point $\left(\sqrt{a}, \sqrt{b}, \sqrt{c}\right)$ is on the sphere if $a + b + c = r^2$. Moreover, by part (a), $abc = f\left(\sqrt{a}, \sqrt{b}, \sqrt{c}\right) \le \left(\frac{r^2}{3}\right)^3 \Rightarrow (abc)^{1/3} \le \frac{r^2}{3} = \frac{a+b+c}{3}$, as claimed.