6.3.1 Answers to Exercise.

1. (a) $\mathbf{v}$ is an eigenvector of A:

   - $A\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0\mathbf{v}$.

   The corresponding eigenvalue is 0.

   (b) $\mathbf{v}$ is the zero column vector.
   It is not an eigenvector of A.

   (c) $\mathbf{v}$ is not an eigenvector of A:

   - $A\mathbf{v} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ is not a scalar multiple of $\mathbf{v}$.

   (d) $\mathbf{v}$ is an eigenvector of A:

   - $A\mathbf{v} = \begin{bmatrix} -3 \\ 0 \\ 3 \\ 3 \\ -3 \end{bmatrix} = -3\mathbf{v}$.

   The corresponding eigenvalue is -3.

2. (a) $p_A(x) = x(x-4)$.
   $\lambda_1 = 0$, $\lambda_2 = 4$.
   An eigenvector of A with eigenvalue λ_1 is given by $\mathbf{u}_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$. Every eigenvector with the same eigenvalue is a non-zero scalar multiple of $\mathbf{u}_1$.
   An eigenvector of A with eigenvalue λ_2 is given by $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Every eigenvector with the same eigenvalue is a non-zero scalar multiple of $\mathbf{u}_2$.
   A is diagonalizable.
   A diagonalization of A is given by $U^{-1}AU = \text{diag}(\lambda_1, \lambda_2)$, in which $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2\ ]$.

   (b) $p_A(x) = (x-3)(x-9)$.
   $\lambda_1 = 3$, $\lambda_2 = 9$.
   An eigenvector of A with eigenvalue λ_1 is given by $\mathbf{u}_1 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$. Every eigenvector with the same eigenvalue is a non-zero scalar multiple of $\mathbf{u}_1$.
   An eigenvector of A with eigenvalue λ_2 is given by $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Every eigenvector with the same eigenvalue is a non-zero scalar multiple of $\mathbf{u}_2$.
   A is diagonalizable.
   A diagonalization of A is given by $U^{-1}AU = \text{diag}(\lambda_1, \lambda_2)$, in which $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2\ ]$.

   (c) $p_A(x) = (x+6)(x-6)$.
   $\lambda_1 = -6$, $\lambda_2 = 6$.
   An eigenvector of A with eigenvalue λ_1 is given by $\mathbf{u}_1 = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$. Every eigenvector with the same eigenvalue is a non-zero scalar multiple of $\mathbf{u}_1$.
   An eigenvector of A with eigenvalue λ_2 is given by $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Every eigenvector with the same eigenvalue is a non-zero scalar multiple of $\mathbf{u}_2$.
   A is diagonalizable.
   A diagonalization of A is given by $U^{-1}AU = \text{diag}(\lambda_1, \lambda_2)$, in which $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2\ ]$.

(d) $p_A(x) = (x+\sqrt{5})(x-\sqrt{5})$.
$\lambda_1 = -\sqrt{5}$, $\lambda_2 = \sqrt{5}$.

An eigenvector of A with eigenvalue λ_1 is given by $\mathbf{u}_1 = \begin{bmatrix} (-3+\sqrt{5})/2 \\ 1 \end{bmatrix}$. Every eigenvector with the same eigenvalue is a non-zero scalar multiple of $\mathbf{u}_1$.

An eigenvector of A with eigenvalue λ_2 is given by $\mathbf{u}_2 = \begin{bmatrix} (-3-\sqrt{5})/2 \\ 1 \end{bmatrix}$. Every eigenvector with the same eigenvalue is a non-zero scalar multiple of $\mathbf{u}_2$.
A is diagonalizable.
A diagonalization of A is given by $U^{-1}AU = \text{diag}(\lambda_1, \lambda_2)$, in which $U = \left[\begin{array}{c|c} \mathbf{u}_1 & \mathbf{u}_2 \end{array}\right]$.

(e) $p_A(x) = -(x+1)(x-1)(x-3)$.
$\lambda_1 = -1$, $\lambda_2 = 1$, $\lambda_3 = 3$.

An eigenvector of A with eigenvalue λ_1 is given by $\mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Every eigenvector with the same eigenvalue is a non-zero scalar multiple of $\mathbf{u}_1$.

An eigenvector of A with eigenvalue λ_2 is given by $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$. Every eigenvector with the same eigenvalue is a non-zero scalar multiple of $\mathbf{u}_2$.

An eigenvector of A with eigenvalue λ_3 is given by $\mathbf{u}_3 = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$. Every eigenvector with the same eigenvalue is a non-zero scalar multiple of $\mathbf{u}_3$.
A is diagonalizable.
A diagonalization of A is given by $U^{-1}AU = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$, in which $U = \left[\begin{array}{c|c|c} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{array}\right]$.

(f) $p_A(x) = -(x+1)(x-1)(x-2)$.
$\lambda_1 = -1$, $\lambda_2 = 1$, $\lambda_3 = 2$.

An eigenvector of A with eigenvalue λ_1 is given by $\mathbf{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Every eigenvector with the same eigenvalue is a non-zero scalar multiple of $\mathbf{u}_1$.

An eigenvector of A with eigenvalue λ_2 is given by $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Every eigenvector with the same eigenvalue is a non-zero scalar multiple of $\mathbf{u}_2$.

An eigenvector of A with eigenvalue λ_3 is given by $\mathbf{u}_3 = \begin{bmatrix} -6/5 \\ -3/5 \\ 1 \end{bmatrix}$. Every eigenvector with the same eigenvalue is a non-zero scalar multiple of $\mathbf{u}_3$.
A is diagonalizable.
A diagonalization of A is given by $U^{-1}AU = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$, in which $U = \left[\begin{array}{c|c|c} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{array}\right]$.

(g) $p_A(x) = -(x-1)(x-2)(x-3)$.
$\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$.

An eigenvector of A with eigenvalue λ_1 is given by $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Every eigenvector with the same eigenvalue is a non-zero scalar multiple of $\mathbf{u}_1$.

An eigenvector of A with eigenvalue λ_2 is given by $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Every eigenvector with the same eigenvalue is a non-zero scalar multiple of $\mathbf{u}_2$.

An eigenvector of A with eigenvalue λ_3 is given by $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Every eigenvector with the same eigenvalue is a non-zero scalar multiple of $\mathbf{u}_3$.
A is diagonalizable.
A diagonalization of A is given by $U^{-1}AU = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$, in which $U = \left[\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3\ \right]$.

(h) $p_A(x) = -(x-1)^2(x-2)$.
$\lambda_1 = 1$, $\lambda_3 = 2$.
Two linearly independent eigenvectors of A with eigenvalue λ_1 are given by $\mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}$. Every eigenvector of A with eigenvalue λ_1 is a linear combination of $\mathbf{u}_1, \mathbf{u}_2$ which is not the zero vector.
An eigenvector of A with eigenvalue λ_3 is given by $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Every eigenvector with the same eigenvalue is a non-zero scalar multiple of $\mathbf{u}_3$.
A is diagonalizable.
A diagonalization of A is given by $U^{-1}AU = \text{diag}(\lambda_1, \lambda_1, \lambda_3)$, in which $U = \left[\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3\ \right]$.

(i) $p_A(x) = -(x-2)^2(x-6)$.
$\lambda_1 = 2$, $\lambda_3 = 6$.
Two linearly independent eigenvectors of A with eigenvalue λ_1 are given by $\mathbf{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Every eigenvector of A with eigenvalue λ_1 is a linear combination of $\mathbf{u}_1, \mathbf{u}_2$ which is not the zero vector.
An eigenvector of A with eigenvalue λ_3 is given by $\mathbf{u}_3 = \begin{bmatrix} 1/3 \\ -2/3 \\ 1 \end{bmatrix}$. Every eigenvector with the same eigenvalue is a non-zero scalar multiple of $\mathbf{u}_3$.
A is diagonalizable.
A diagonalization of A is given by $U^{-1}AU = \text{diag}(\lambda_1, \lambda_1, \lambda_3)$, in which $U = \left[\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3\ \right]$.

(j) $p_A(x) = -(x+1)^2(x-3)$.
$\lambda_1 = -1$, $\lambda_3 = 3$.
Two linearly independent eigenvectors of A with eigenvalue λ_1 are given by $\mathbf{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Every eigenvector of A with eigenvalue λ_1 is a linear combination of $\mathbf{u}_1, \mathbf{u}_2$ which is not the zero vector.
An eigenvector of A with eigenvalue λ_3 is given by $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. Every eigenvector with the same eigenvalue is a non-zero scalar multiple of $\mathbf{u}_3$.
A is diagonalizable.
A diagonalization of A is given by $U^{-1}AU = \text{diag}(\lambda_1, \lambda_1, \lambda_3)$, in which $U = \left[\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3\ \right]$.

(k) $p_A(x) = (x+9)(x+4)(x-4)(x-9)$.
$\lambda_1 = -9$, $\lambda_2 = -4$, $\lambda_3 = 4$, $\lambda_3 = 9$.
An eigenvector of A with eigenvalue λ_1 is given by $\mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ -3 \\ 0 \end{bmatrix}$. Every eigenvector with the same eigenvalue is a non-zero scalar multiple of $\mathbf{u}_1$.

An eigenvector of A with eigenvalue λ_2 is given by $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -4 \end{bmatrix}$. Every eigenvector with the same eigenvalue is a non-zero scalar multiple of $\mathbf{u}_2$.

An eigenvector of A with eigenvalue λ_3 is given by $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 4 \end{bmatrix}$. Every eigenvector with the same eigenvalue is a non-zero scalar multiple of $\mathbf{u}_3$.

An eigenvector of A with eigenvalue λ_4 is given by $\mathbf{u}_4 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}$. Every eigenvector with the same eigenvalue is a non-zero scalar multiple of $\mathbf{u}_4$.
A is diagonalizable.

(l) $p_A(x) = (x-1)^3(x-2)$.
$\lambda_1 = 1$, $\lambda_2 = 2$.

Three linearly independent eigenvectors of A with eigenvalue λ_1 is given by $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. Every eigenvector with eigenvalue λ_1 is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ whis is not the zero vector.

An eigenvector of A with eigenvalue λ_2 is given by $\mathbf{u}_2 = \begin{bmatrix} -1/2 \\ -1/2 \\ 0 \\ 1 \end{bmatrix}$. Every eigenvector with the same eigenvalue is a non-zero scalar multiple of $\mathbf{u}_2$.
A is diagonalizable.
A diagonalization of A is given by $U^{-1}AU = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, in which $U = \left[\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4\ \right]$.

3. (a) $p_A(x) = x^2 - 4x + 8$.
   (b) The eigenvalues of A are $\lambda_1 = 2 + 2i$, $\lambda_2 = 2 - 2i$.
   (c) An eigenvector of A with eigenvalue λ_1 is given by $\mathbf{u}_1 = \begin{bmatrix} -1-2i \\ 1 \end{bmatrix}$. Every eigenvector of A with eigenvalue λ_1 is a non-zero scalar multiple of $\mathbf{u}_1$.
   An eigenvector of A with eigenvalue λ_2 is given by $\mathbf{u}_2 = \begin{bmatrix} -1+2i \\ 1 \end{bmatrix}$. Every eigenvector of A with eigenvalue λ_2 is a non-zero scalar multiple of $\mathbf{u}_2$.
   (d) Note that $\mathbf{u}_1, \mathbf{u}_2$ are two linearly independent eigenvectors of A (as they are corresponding to distinct eigenvalues of A).
   Also note that A is a (2×2)-square matrix.
   Then A is diagonalizable.
   A diagonalization of A is given by $U^{-1}AU = \text{diag}(\lambda_1, \lambda_2)$, in which $U = \left[\ \mathbf{u}_1 \mid \mathbf{u}_2\ \right]$.

4. ——

5. (a) $p_A(x) = (x-1)^2$.
   (b) The only eigenvalue of A is $\lambda_1 = 1$.
   (c) An eigenvector of A with eigenvalue λ_1 is given by $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Every eigenvector of A with eigenvalue λ_1 is a non-zero scalar multiple of $\mathbf{u}_1$.

(d) A is not diagonalizable.

Reason: In order for A to be diagonalizable, it is necessary for a pair of linearly independent eigenvectors of A to be available. However, every eigenvector of A is a non-zero scalar multiple of $\mathbf{u}_1$.

6. (a) $p_A(x) - (x-2)^2(x+5)$.

(b) The eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = -5$.

(c) An eigenvector of A with eigenvalue λ_1 is given by $\mathbf{u}_1 = \begin{bmatrix} -10 \\ 3 \\ 1 \end{bmatrix}$.

Every eigenvector of A with eigenvalue λ_1 is a non-zero scalar multiple of $\mathbf{u}_1$.

An eigenvector of A with eigenvalue λ_2 is given by $\mathbf{u}_2 = \begin{bmatrix} 4 \\ -4 \\ 1 \end{bmatrix}$.

Every eigenvector of A with eigenvalue λ_2 is a non-zero scalar multiple of $\mathbf{u}_2$.

(d) A is not diagonalizable.

7. (a) The only eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 2$.

(b)
- Two linearly independent eigenvectors of A with eigenvalue λ_1 are given by $\mathbf{u}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 \begin{bmatrix} 1/2 \\ -7/4 \\ 0 \\ -3/4 \\ 1 \end{bmatrix}$.

  Every eigenvector of A with eigenvalue λ_1 is a linear combination of $\mathbf{u}_1, \mathbf{u}_2$ which is not the zero vector.
- Two linearly independent eigenvectors of A with eigenvalue λ_2 are given by $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -8 \\ 5 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_4 \begin{bmatrix} 1 \\ -10 \\ 6 \\ 0 \\ 1 \end{bmatrix}$.

  Every eigenvector of A with eigenvalue λ_1 is a linear combination of $\mathbf{u}_1, \mathbf{u}_2$ which is not the zero vector.

(c) A is not diagonalizable.

Comment.
In order for A to be diagonalizable, it is necessary and sufficient for there to be five linearly independent eigenvectors of A.
Suppose there were five linearly independent eigenvectors of A, say, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$.
Without loss of generality, suppose the eigenvalue of $\mathbf{v}_1$ is λ_1. Then amongst $\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$, there would be at most one of them of eigenvalue λ_1. (If there were two of them, say, $\mathbf{v}_2, \mathbf{v}_3$, then $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ would be three linearly independent column vectors belonging to $\mathcal{N}(A - \lambda_1 I_5)$, which is of dimension 2.)
Therefore, at least three of $\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ would have eigenvalue λ_2. Then $\mathcal{N}(A - \lambda_2 I_5)$ would be of dimension at least 3.
But $\mathcal{N}(A - \lambda_2 I_5)$ is of dimension 2 only.

8. (a) $p_{A_\alpha}(x) = (1-x)(3-x)^2$.

(b) The eigenvalues of A_α are $1, 3$.

(c)
- (Case 1). Suppose $\alpha \neq 1$ and $\alpha \neq -1$ and $\alpha \neq 2$.

  An eigenvector of A_α with eigenvalue 1 is given by $\mathbf{u}_1 = \begin{bmatrix} -2/[(\alpha+1)(\alpha-2)] \\ -2\alpha/[(\alpha+1)(\alpha-2)] \\ 1 \end{bmatrix}$.

  Every eigenvector of A_α with eigenvalue 1 is a non-zero scalar multiple of $\mathbf{u}_1$.

  An eigenvector of A_α with eigenvalue 3 is $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

  Every eigenvector of A_α with eigenvalue 3 is a non-zero scalar multiple of $\mathbf{u}_2$.

- (Case 2).

  An eigenvector of A_{-1} with eigenvalue 1 is given by $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

  Every eigenvector of A_{-1} with eigenvalue 1 is a non-zero scalar multiple of $\mathbf{u}_1$.

  An eigenvector of A_{-1} with eigenvalue 3 is $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

  Every eigenvector of A_{-1} with eigenvalue 3 is a non-zero scalar multiple of $\mathbf{u}_2$.
- (Case 3).

  An eigenvector of A_1 with eigenvalue 1 is given by $\mathbf{u}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

  Every eigenvector of A_1 with eigenvalue 1 is a non-zero scalar multiple of $\mathbf{u}_1$.

  Two linearly indepdent eigenvector of A_1 with eigenvalue 3 are given by $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

  Every eigenvector of A_1 with eigenvalue 3 is a linear combination of $\mathbf{u}_2, \mathbf{u}_3$ which is not the zero vector.
- (Case 4).

  An eigenvector of A_2 with eigenvalue 1 is given by $\mathbf{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

  Every eigenvector of A_2 with eigenvalue 1 is a non-zero scalar multiple of $\mathbf{u}_1$.

  An eigenvector of A_2 with eigenvalue 3 is $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

  Every eigenvector of A_2 with eigenvalue 3 is a non-zero scalar multiple of $\mathbf{u}_2$.

(d) A_α is diagonalizable if and only if $\alpha = 1$.

9. (a) The eigenvalues of A are $\lambda_1 = -1$, $\lambda_2 = 1$, $\lambda_3 = 2$.

   (b)
   - An eigenvector of A with eigenvalue λ_1 is given by $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.
   - Two linearly independent eigenvectors of A with eigenvalue λ_2 are given by $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 3 \\ -1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.
   - Two linearly independent eigenvectors of A with eigenvalue λ_3 are given by $\mathbf{u}_4 = \begin{bmatrix} 1/2 \\ -1/2 \\ 3/2 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_5 = \begin{bmatrix} 1/2 \\ 1/2 \\ -5/2 \\ 0 \\ 1 \end{bmatrix}$.

   (c) A is diagonalizable because A is a (5×5)-square matrix and $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$ are five linearly independent eigenvectors of A.
   A diagonalization of A is given by $U^{-1}AU = E$, in which $U = \left[\begin{array}{c|c|c|c|c} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 & \mathbf{u}_5 \end{array}\right]$, and $E = \mathrm{diag}(-1, 1, 1, 2, 2)$.

   (d) i. The eigenvalues of A^2 are $\mu_1 = 1$, $\mu_2 = 4$.

   ii. The characteristic polynomial $p_{A^2}(x)$ of A^2 is given by $p_{A^2}(x) = -(x-1)^3(x-4)^2$.

   iii. • Three linearly independent eigenvectors of A with eigenvalue μ_1 are given by $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 3 \\ -1 \\ 1 \\ 0 \end{bmatrix}$,

$$\mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

- Two linearly independent eigenvectors of A with eigenvalue μ_2 are given by $\mathbf{u}_4 = \begin{bmatrix} 1/2 \\ -1/2 \\ 3/2 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_5 = \begin{bmatrix} 1/2 \\ 1/2 \\ -5/2 \\ 0 \\ 1 \end{bmatrix}$.

iv. A diagonalization of A^2 is given by $U^{-1}A^2U = E^2$, in which $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4 \mid \mathbf{u}_5\ ]$, and $E^2 = \mathrm{diag}(1, 1, 1, 4, 4)$.

10. ——

11. (a) $c_0 = ad - bc$, $c_1 = -(a + d)$, $c_2 = 1$.

(b) ——

(c) i. $p_A(x) = (x - \lambda)^2$; $c_0 = \lambda^2$, $c_1 = -2\lambda$, $c_2 = 1$.

ii. $A^2 = -\lambda^2 I_2 + 2\lambda A$.
$A^3 = -2\lambda^3 I_2 + 3\lambda^2 A$.
$A^4 = -3\lambda^4 I_2 + 4\lambda^3 A$.
$A^5 = -4\lambda^5 I_2 + 5\lambda^4 A$.

iii. Whenever n is an integer greater than 1, the equality $A^n = -(n-1)\lambda^n I_2 + n\lambda^{n-1}A$ holds.

12. (a) $p_A(x) = (x - 1)^3(x - 2)$ as polynomials.

(b) $1, 2$ are the only eigenvalues of A.

(c)
- Write $\lambda_1 = 1$.
The eigenspace of A corresponding to λ_1 is $\mathcal{N}(A - \lambda_1 I_2)$.
It is of dimension 1, and a basis for it is given by $\mathbf{v}_1 = \begin{bmatrix} -3/5 \\ -6/5 \\ 4/5 \\ 1 \end{bmatrix}$.
- Write $\lambda_2 = 2$.
The eigenspace of A corresponding to λ_2 is $\mathcal{N}(A - \lambda_2 I_2)$.
It is of dimension 1, and a basis for it is given by $\mathbf{v}_2 = \begin{bmatrix} -2/3 \\ -1 \\ 2/3 \\ 1 \end{bmatrix}$.

(d) A is not diagonalizable.

13. (a) *Comment.*
A is a (5×5)-square matrix, and has five pairwise distinct eigenvalues, namely, $-2, -\sqrt{3}, 0, \sqrt{3}, 2$.

(b) $p_A(x) = -12x + 7x^3 - x^5$.

(c) i. $A^5 = 7A^3 - 12A$.

ii. $A^{10} = 175A^4 - 444A^2$.

14. (a) i. $\alpha = 2, \beta = 8$.

ii. $p_A(x) = x^2 - 2x - 8$.
The eigenvalues of A are $-2, 4$.

iii. A diagonalization of A is given by $U^{-1}AU = \mathrm{diag}(4, -2)$, in which $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2\ ]$, $\mathbf{u}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.
$\lambda = 4$, $\mu = -2$.

iv. $x_n = \dfrac{4^n}{6} + (-1)^{n-1} \cdot \dfrac{2^{n-1}}{3}$ for each natural number n.

(b) i. $x_n = 2(-1)^{n+1} + 2^{n+1}$ for each natural number n.

ii. $x_n = -2^{n+2} + 3^{n+1}$ for each natural number n.

iii. $x_n = 1 + 2^n + 3^n$ for each natural number n.

15. (a) $\mathbf{x}$ is an eigenvector of B with eigenvalue $c_0 + c_1\lambda + c_2\lambda^2 + c_3\lambda^3$.

(b) i. ——

ii. $\mathbf{y}$ is an eigenvector of C with eigenvalue $\lambda + \lambda^{-1}$.

16. ——

17. (a) True.

(b) False. One possible choice of counter-examples is $A = I_2$, $\lambda = 1$.

18. ——

19. ——

20. ——

21. ——

22. ——

23. (a) ——

(b) ——

(c) $\mathcal{LS}(T - \lambda I_6, \mathbf{z})$ is inconsistent.

(d) $\mathcal{LS}(H - \mu I_6, \mathbf{y})$ is inconsistent.