6.3.1 Exercise: Eigenvalues, eigenvectors, and diagonalization.

1. Consider each pairs of square matrices and column vectors below, respectively labelled $A, \mathbf{v}$ here.
   Determine whether $\mathbf{v}$ is an eigenvector of A. If it is, also determine the corresponding eigenvalue.

(a) $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & -1 & 5 \\ 0 & 1 & 1 & 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

(c) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 1 & -1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

(d) $A = \begin{bmatrix} -13 & -25 & 3 & -17 & -4 \\ 6 & 11 & 0 & 8 & 2 \\ 7 & 13 & -3 & 8 & 1 \\ 5 & 10 & -3 & 6 & 1 \\ -11 & -20 & 3 & -13 & -2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}$.

2. For the square matrix with real entries in each part below, denoted by A:—
   - determine its characteristic polynomial $p_A(x)$,
   - determine its eigenvalues $\lambda_1, \lambda_2, \cdots$, and
   - determine all eigenvectors corresponding to the eigenvalue λ_j for each j, and
   - provide a diagonalization for A with respect to some appropriate invertible square matrix, if A is diagonalizable.

(a) $A = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$

(b) $A = \begin{bmatrix} 5 & 8 \\ 1 & 7 \end{bmatrix}$

(c) $A = \begin{bmatrix} -8 & 7 \\ -4 & 8 \end{bmatrix}$

(d) $A = \begin{bmatrix} 3 & 2 \\ -2 & -3 \end{bmatrix}$

(e) $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & -3 \\ 1 & -1 & 0 \end{bmatrix}$

(f) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ -2 & -1 & -1 \end{bmatrix}$

(g) $A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

(h) $A = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$

(i) $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & -2 \\ -3 & -3 & 5 \end{bmatrix}$

(j) $A = \begin{bmatrix} -3 & -2 & 2 \\ 4 & 3 & -4 \\ -2 & -2 & 1 \end{bmatrix}$

(k) $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 27 & 0 & 0 \\ 16 & 0 & 0 & 0 \end{bmatrix}$

(l) $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 3 \end{bmatrix}$

3. Let $A = \begin{bmatrix} 3 & 5 \\ -1 & 1 \end{bmatrix}$.
   (a) Find the characteristic polynomial $p_A(x)$ of A.
   (b) Write down the eigenvalues of A.
   (c) For each eigenvalue of A, determine all its corresponding eigenvectors.
   (d) Is A diagonalizable? Justify your answer. If A is diagonalizable, also give a diagonalization of A with respect to some appropriate invertible matrix.

4. With direct reference to the definition for the notion of eigenvalues, eigenvectors, and linear dependence/independence, prove the statements below:
   (a) *Let A be an $(n \times n)$-square matrix, λ_1, λ_2 be pairwise distinct numbers, and $\mathbf{u}_1, \mathbf{u}_2$ be non-zero column vectors with n entries.*
   Suppose $\mathbf{u}_1, \mathbf{u}_2$ are eigenvectors of A with eigenvalues λ_1, λ_2 respectively. Then $\mathbf{u}_1, \mathbf{u}_2$ are linearly independent.
   (b) *Let A be an $(n \times n)$-square matrix, $\lambda_1, \lambda_2, \lambda_3$ be pairwise distinct numbers, and $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be non-zero column vectors with n entries.*
   Suppose $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are eigenvectors of A with eigenvalues $\lambda_1, \lambda_2, \lambda_3$ respectively. Then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly independent.
   Remark. You may find that it is better to make use of the result in the previous part in an appropriate way.
   (c) *Let A be an $(n \times n)$-square matrix, $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be pairwise distinct numbers, and $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ be non-zero column vectors with n entries.*
   Suppose $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are eigenvectors of A with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ respectively. Then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are linearly independent.
   Remark. You may find that it is better to make use of the result in the previous part in an appropriate way.

5. Let $A = \begin{bmatrix} 5 & -4 \\ 4 & -3 \end{bmatrix}$.

(a) Find the characteristic polynomial $p_A(x)$ of A.

(b) Write down the eigenvalues of A.

(c) For each eigenvalue of A, determine all its corresponding eigenvectors.

(d) Is A diagonalizable? Justify your answer.

6. Let $A = \begin{bmatrix} 0 & 0 & -20 \\ 1 & 0 & 16 \\ 0 & 1 & -1 \end{bmatrix}$.

(a) Find the characteristic polynomial $p_A(x)$ of A.

(b) Write down the eigenvalues of A.

(c) For each eigenvalue of A, determine all its corresponding eigenvectors.

(d) Is A diagonalizable? Justify your answer.

7. Let $A = \begin{bmatrix} 0 & -2 & -2 & -4 & -6 \\ 16 & 22 & 21 & 39 & 58 \\ -9 & -12 & -11 & -22 & -33 \\ 3 & 4 & 4 & 11 & 13 \\ -4 & -5 & -5 & -11 & -14 \end{bmatrix}$.

Take for granted that the characteristic polynomial $p_A(x)$ of the matrix A is given by $p_A(x) = -(x-1)^2(x-2)^3$. Further take for granted that $A - I_5$ is row-equivalent to

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 1 & 0 & \frac{7}{4} \\ 0 & 0 & 0 & 1 & \frac{3}{4} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and $A - 2I_5$ is row-equivalent to

$$C = \begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 8 & 10 \\ 0 & 0 & 1 & -5 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Name all possible eigenvalues of A.

(b) Determine all eigenvectors of A.

(c) Is A diagonalizable? Justify your answer.

8. Let α be a number, and $A_\alpha = \begin{bmatrix} 1 & 0 & 0 \\ \alpha & 3 & 0 \\ 1 & \alpha - 1 & 3 \end{bmatrix}$.

(a) Write down the characteristic polynomial $p_{A_\alpha}(x)$ of the matrix A_α.

(b) What are the eigenvalues of A_α?

(c) For each value of α, determine all possible eigenvectors of A_α.

(d) For which values of α is A diagonalizable? Justify your answer.

9. Let $A = \begin{bmatrix} 41 & 2 & -5 & -11 & -33 \\ 33 & 3 & -4 & -10 & -27 \\ -2 & 0 & 2 & 1 & 1 \\ 41 & 2 & -5 & -10 & -34 \\ 39 & 2 & -5 & -11 & -31 \end{bmatrix}$.

Take for granted that the characteristic polynomial $p_A(x)$ of the matrix A is given by $p_A(x) = -(x-1)^2(x-2)^2(x+1)$. Further take for granted that:—

- $A + I_5$ is row-equivalent to $B = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$,
- $A - I_5$ is row-equivalent to $C = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, and
- $A - 2I_5$ is row-equivalent to $D = \begin{bmatrix} 1 & 0 & 0 & -1/2 & -1/2 \\ 0 & 1 & 0 & 1/2 & -1/2 \\ 0 & 0 & 1 & -3/2 & 5/2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Name all possible eigenvalues of A.

(b) Determine all eigenvectors of A.

(c) Show that A is diagonalizable. Also give a diagonalization of A.

(d) i. Name all possible eigenvalues of A^2.

ii. Write down the characteristic polynomial $p_{A^2}(x)$ of the matrix A^2.

iii. Determine all eigenvectors of A^2.

iv. Write down a diagonalization of A^2.

10. (a) Prove the statement (♯):—

(♯) *Let a_1, a_2, c be real numbers. Suppose $A = \begin{bmatrix} a_1 & c \\ c & a_2 \end{bmatrix}$, and $\alpha = \dfrac{a_1 + a_2}{2}$, and $\beta = \dfrac{a_1 - a_2}{2}$.*
Then $p_A(x) = (x-\alpha)^2 - (\beta^2 + c^2)$ as polynomials.

(b) Hence, or otherwise, prove that the statement (♮):

(♮) Let A be a (2×2)-square matrix with real entries. Suppose A is symmetric. Then the eigenvalues of A are real. Moreover A is diagonalizable with respect to some invertible (2×2)-square matrix with real entries.

11. Let a, b, c, d be numbers, and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Write $p_A(x) = c_0 + c_1 x + c_2 x^2$. Here c_0, c_1, c_2 stand for some appropriate numbers.

(a) Write down the values c_0, c_1, c_2. Leave your answers in terms of a, b, c, d.

(b) By direct calculation, verify that $c_0 I_2 + c_1 A + c_2 A^2 = \mathcal{O}_{2\times 2}$.

(c) Suppose A has one and only one eigenvalue, say, λ.

i. Express $p_A(x)$ in terms of λ. Give the values of c_0, c_1, c_2 in terms of λ.

ii. Express A^2, A^3, A^4, A^5 in terms of λ, I_2, A.

iii. Conjecture a 'general formula' for A^n in terms of λ, I_2, A that holds whenever n is an integer greater than 1. Prove your conjecture with mathematical induction.

12. Let $A = \begin{bmatrix} 1 & -4 & -1 & -4 \\ 2 & 0 & 5 & -4 \\ -1 & 1 & -2 & 3 \\ -1 & 4 & -1 & 6 \end{bmatrix}$.

Take for granted its characteristic polynomial $p_A(x)$ is given by $p_A(x) = x^4 - 5x^3 + 9x^2 - 7x + 2$, and $1, 2$ are some of the roots of $p_A(x)$.

(a) Factorize $p_A(x)$ completely.

(b) What are the eigenvalues of A? Justify your answer.

(c) Determine the eigenspaces of A explicitly, by naming a basis for each of them.

(d) Is A diagonalizable? Justify your answer.

13. Let A be a (5×5)-square matrix.

Suppose A is not invertible, and $\det(A + 2I_5) = 0$, $\det(A + \sqrt{3}I_5) = 0$, $\det(A - \sqrt{3}I_5) = 0$ and $\det(A - 2I_5) = 0$.

(a) Explain why A is diagonalizable.

(b) Write down the characteristic polynomial $p_A(x)$.
Arrange the terms $p_A(x)$ in ascending powers of x.

(c) i. By using the diagonalizability of A, or otherwise, show that $A^5 = \alpha A^3 + \beta A$.
Here α, β are some appropriate numbers. You have to give the value of α, β explicitly.

ii. Express A^{10} as a sum of scalar multiples of I_5, A, A^2, A^3, A^4.

14. (a) Let $\{x_n\}_{n=0}^{\infty}$ be the infinite sequence of real numbers defined recursively by

$$\begin{cases} x_0 &= 0 \\ x_1 &= 1 \\ x_{n+2} &= 2x_{n+1} + 8x_n \quad \text{for any natural number } n \end{cases}$$

i. Write down the (2×2)-square matrix $A = \begin{bmatrix} \alpha & \beta \\ 1 & 0 \end{bmatrix}$ for which the equality $\begin{bmatrix} x_{n+2} \\ x_{n+1} \end{bmatrix} = A \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$ holds for every natural number n.
Here α, β are some appropriate real numbers, independent of n, which you have to determine explicitly.

ii. Find the characteristic polynomial $p_A(x)$, and find the eigenvalues of A.

iii. Hence find a diagaonalization for A, and show that $A^n = \frac{\lambda^n}{6}\begin{bmatrix} 4 & 8 \\ 1 & 2 \end{bmatrix} + \frac{\mu^n}{6}\begin{bmatrix} 2 & -8 \\ -1 & 4 \end{bmatrix}$ for every natural number n.
Here λ, μ are some appropriate real numbers, independent of n, which you have to determine explicitly.

iv. Hence find an explicit formula for x_n (in terms of n alone).

(b) Imitate the process described above to find an explicit formula for the individual terms of each recursively defined infinite sequence described below:

i. $$\begin{cases} x_0 &= 0 \\ x_1 &= 6 \\ x_{n+2} &= x_{n+1} + 2x_n \quad \text{for any natural number } n \end{cases}$$

ii. $$\begin{cases} x_0 &= -1 \\ x_1 &= 1 \\ x_{n+2} &= 5x_{n+1} - 6x_n \quad \text{for any natural number } n \end{cases}$$

iii. $$\begin{cases} x_0 &= 3 \\ x_1 &= 6 \\ x_2 &= 14 \\ x_{n+3} &= 6x_{n+2} - 11x_{n+1} + 6x_n \quad \text{for any natural number } n \end{cases}$$

15. Let A be an $(n \times n)$-square matrix, and λ be a number. Suppose $\mathbf{x}$ is an eigenvector of A with eigenvalue λ.

(a) Let c_0, c_1, c_2, c_3 be numbers, and $B = c_0 I_n + c_1 A + c_2 A^2 + c_3 A^3$.
Is $\mathbf{x}$ an eigenvector of B? Justify your answer, with direct reference to the definition of eigenvalues and eigenvectors.

(b) Now suppose A is invertible.

i. Show that $\lambda \neq 0$, and $\mathbf{x}$ is an eigenvector of A^{-1} with eigenvalue λ^{-1}.

ii. Define the $(2n \times 2n)$-square matrices C and the column vector $\mathbf{y}$ with $2n$ entries by

$$C = \left[\begin{array}{c|c} A & A^{-1} \\ \hline A^{-1} & A \end{array}\right], \qquad \mathbf{y} = \left[\begin{array}{c} \mathbf{x} \\ \hline \mathbf{x} \end{array}\right]$$

Is $\mathbf{y}$ an eigenvector of C? Justify your answer with direct reference to the definition of eigenvalues and eigenvectors.

16. Prove the statements below with reference to the definitions for the notions of *eigenvalues*, *eigenvectors*, *diagonalizability*, *integral powers of (square) matrices* and *invertibility*, where relevant:—

(a) i. *Let A be a square matrix. Suppose λ is an eigenvalue of A, and $\mathbf{v}$ is an eigenvector of A with eigenvalue λ. Then for each positive integer p, λ^p is an eigenvalue of A, and $\mathbf{v}$ is an eigenvector of A^p with eigenvalue λ^p.*

ii. *Let A be a square matrix. Suppose A is diagonalizable, with respect to some invertible matrix U of the same size. Then, for any positive integer p, A^p is diagonalizable with respect to the same U.*

(b) i. *Let A be a square matrix. Suppose A is invertible. Further suppose λ is an eigenvalue of A, and $\mathbf{v}$ is an eigenvector of A with eigenvalue λ. Then $\lambda \neq 0$. Moreover, for each positive integer q, λ^{-q} is an eigenvalue of A^{-q}, and $\mathbf{v}$ is an eigenvector of A^{-q} with eigenvalue λ^{-q}.*

ii. *Let A be a square matrix. Suppose A is invertible. Also suppose A is diagonalizable, with respect to some invertible matrix U of the same size. Then for each positive integer q, A^{-q} is diagonalizable with respect to the same U.*

17. For each statement below, determine whether it true or false. Justify your answer with an appropriate argument.

(a) *Let A be an $(n \times n)$-square matrix, and λ be a number.*
Suppose λ^2 is an eigenvalue of A^2. Then at least one of $\lambda, -\lambda$ is an eigenvalue of A.

(b) *Let A be an $(n \times n)$-square matrix, and λ be a number.*
Suppose λ^2 is an eigenvalue of A^2. Then both of $\lambda, -\lambda$ is an eigenvalue of A.

18. Let A be a square matrix, and α, β be numbers. Prove the statements below:—

(a) *$\alpha\beta$ is an eigenvalue of $(\alpha + \beta)A - A^2$ if and only if at least one of α, β is an eigenvalue of A.*

(b) *Suppose A is invertible.*
Then $\alpha + \beta$ is an eigenvalue of $A + \alpha\beta A^{-1}$ if and only if at least one of α, β is an eigenvalue of A.

19. Prove the statements below with reference to the definitions for the notions of *eigenvalues*, *eigenvectors*, *diagonalizability*, where relevant:—

(a) *Let A, B be $(n \times n)$-square matrices, λ, μ be numbers, and $\mathbf{v}$ be a non-zero column vector with n entries.*
Suppose $\mathbf{v}$ is an eigenvector of A, B with respective eigenvalues λ, μ.
Then for any numbers α, β, $\mathbf{v}$ is an eigenvector of $\alpha A + \beta B$ with eigenvalue $\alpha\lambda + \beta\mu$.

(b) *Let A, B, U be $(n \times n)$-square matrices. Suppose U is invertible, and suppose A, B are diagonalizable with respect to U.*
Then for any numbers α, β, $\alpha A + \beta B$ is diagonalizable with respect to U.

20. Prove the statements below with reference to the definitions for the notions of *eigenvalues*, *eigenvectors*, *diagonalizability*, where relevant:—

(a) *Let A, B be $(n \times n)$-square matrices, λ, μ be numbers, and $\mathbf{v}$ be a non-zero column vector with n entries.*
Suppose $\mathbf{v}$ is an eigenvector of A, B with respective eigenvalues λ, μ.
Then $\mathbf{v}$ is an eigenvector of each of AB, BA, with eigenvalue $\lambda\mu$ for each.

(b) *Let A, B, U be $(n \times n)$-square matrices. Suppose U is invertible, and suppose A, B are diagonalizable with respect to U.*
Then AB is diagonalizable with respect to U, and A, B commute with each other.

21. Prove the statements below:—

(a) *Let A, B be $(n \times n)$-square matrices. Suppose A is diagonalizable.*

Further suppose every eigenvector of A is an eigenvector of B.

Then A, B commute with each other.

(b) *Let A, B be $(n \times n)$-square matrices. Suppose A, B commute with each other.*

Further suppose A has n pairwise distinct eigenvalues, and B is invertible. Then every eigenvector of A is an eigenvector of B, and B is diagonalizable.

22. Let A, B be $(n \times n)$-square matrices.

Prove the statements below with reference to the definition for the notions of *eigenvalues* and *eigenvectors*:

(a) *Suppose 0 is an eigenvalue of AB. Then 0 is an eigenvalue of BA.*

(b) *Suppose λ is a non-zero number. Suppose λ is an eigenvalue of AB. Then λ is an eigenvalue of BA.*

23. (a) Prove the statement ($\sharp$):—

($\sharp$) *Let $\mathbf{w}$ be a column vector with 6 real entries. Suppose $\mathbf{w}^t\mathbf{w} = 0$. Then $\mathbf{w} = \mathbf{0}_6$.*

(b) Prove the statement ($\natural$):——

($\natural$) *Suppose S is a (6×6)-symmetric matrix with real entries. Then, for any column vectors $\mathbf{u}, \mathbf{v}$ with 6 real entries, if $\mathbf{u} \in \mathcal{N}(S)$ and $\mathbf{v} \in \mathcal{C}(S)$ then $\mathbf{u}^t\mathbf{v} = 0$.*

(c) Let T be a (6×6)-symmetric matrix with real entries, λ be a real number, and $\mathbf{z}$ be a non-zero column vector with 6 real entries. Suppose $\mathbf{z}$ is an eigenvector of T with eigenvalue λ.

Is the system $\mathcal{LS}(T - \lambda I_6, \mathbf{z})$ consistent or not? Justify your answer.

(d) Let H be a (6×6)-skew-symmetric matrix with real entries, μ be a non-zero real number, and $\mathbf{y}$ be a non-zero column vector with 6 real entries. Suppose $\mathbf{y}$ is an eigenvector of H with eigenvalue μ.

Is the system $\mathcal{LS}(H - \mu I_6, \mathbf{y})$ consistent or not? Justify your answer.