6.1 Eigenvalues and eigenvectors for square matrices.

0. *Assumed background.*

- What has been covered in Topics 1-5, especially:—
  - ∗ 3.3 *Various necessary and sufficient conditions for invertibility.*
  - ∗ 4.4 *Basis for subspaces of column vectors.*
  - ∗ 4.5 *Dimension for subspaces of column vectors.*
  - ∗ 4.7 *Basis and dimension, null space, and the notion of subspace of a subspace.*
  - ∗ 5.2 *Row operations and determinants.*

Abstract. We introduce:—

- the notions of eigenvalues and eigenvectors for square matrices, and
- various equivalent formulations for the notion of eigenvalues.

1. **Definition. (Eigenvalues and eigenvectors.)**

Let A be an $(n \times n)$-square matrix. Let λ be a number. Let $\mathbf{v}$ be a non-zero column vector with n entries.

We say $\mathbf{v}$ is an **eigenvector of A with eigenvalue** λ *if and only if $A\mathbf{v} = \lambda\mathbf{v}$.*

In this situation, we may equivalently say that λ is an **eigenvalue of A with a corresponding eigenvector $\mathbf{v}$**.

Further remarks on terminologies.

(a) We may write

‘*the number λ is an eigenvalue of A*’

(without mentioning any specific corresponding eigenvector) exactly when there is some non-zero column vector $\mathbf{u}$ so that $\mathbf{u}$ is an eigenvector of A with eigenvalue λ.

(b) We may write

‘*the non-zero column vector $\mathbf{v}$ is an eigenvector of A*’

(without mentioning its corresponding eigenvalue) exactly when the equality $A\mathbf{v} = \mu\mathbf{v}$ holds for some number μ.

2. We state a few basic results about eigenvalues and eigenvectors that follow immediately from definition. Their proofs are left as exercises.

Theorem (1). (Uniqueness of eigenvalue corresponding to the same eigenvector.)

Let A be an $(n \times n)$-square matrix.

Let $\mathbf{v}$ be a non-zero column vector with n real entries. Let λ, μ be numbers.

Suppose $\mathbf{v}$ is an eigenvector of A with eigenvalue λ and also with eigenvalue μ.

Then $\lambda = \mu$.

Remark. In plain words, every eigenvector of A corresponds to a unique eigenvalue.

So it makes sense to write ‘*λ is the eigenvalue of A corresponding to eigenvector $\mathbf{v}$*’.

3. **Theorem (2). (Non-zero scalar multiples of eigenvectors.)**

Let A be an $(n \times n)$-square matrix.

Let $\mathbf{v}$ be a non-zero column vector with n entries. Let λ be a number.
Suppose $\mathbf{v}$ is an eigenvector of A with eigenvalue λ.

Then, every non-zero scalar multiple of $\mathbf{v}$ is an eigenvector of A with eigenvalue λ.

Remark. We have not ruled out the possibility that non-zero column vectors which are not scalar multiples of each other can be eigenvectors of A with the same eigenvalue.

4. Theorem (2) is superceded by Theorem (3).

Theorem (3). (Linear combinations of eigenvectors with common eigenvalue.)

Let A be an $(n \times n)$-square matrix.

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ be non-zero column vectors with n entries.

Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ are eigenvectors of A with a common eigenvalue λ.

Then every linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$, except the zero column vector with n entries, is also an eigenvector of A with eigenvalue λ.

Remarks.

(a) As shall be seen in Example (1), it is possible for a square matrix to have two eigenvectors which are not scalar multiples of each other but which correspond to the same eigenvalue.

(b) At this moment, we deliberately refrain from saying anything of linear combinations of eigenvectors of A with distinct eigenvalues.

5. **Example (1). (Eigenvalues and eigenvectors.)**

(a) Let $A = \begin{bmatrix} 13 & 30 \\ -6 & -14 \end{bmatrix}$, and $\mathbf{u}_1 = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Note that neither $\mathbf{u}_1, \mathbf{u}_2$ is the zero column vector.

i. We have $A\mathbf{u}_1 = \begin{bmatrix} 5 \\ -2 \end{bmatrix} = 1 \cdot \mathbf{u}_1$.

Then $\mathbf{u}_1$ is an eigenvector of A with eigenvalue $\lambda_1 = 1$.
Every non-zero scalar multiple of $\mathbf{u}_1$ is also an eigenvector of A with eigenvalue λ_1.

ii. We have $A\mathbf{u}_2 = \begin{bmatrix} -4 \\ 2 \end{bmatrix} = -2\mathbf{u}_2$.

Then $\mathbf{u}_2$ is an eigenvector of A with eigenvalue $\lambda_2 = -2$.
Every non-zero scalar multiple of $\mathbf{u}_2$ is also an eigenvector of A with eigenvalue λ_2.

(b) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, and $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$.

Note that none of $\mathbf{u}_1, \mathbf{u}_1, \mathbf{u}_3$ is the zero column vector.

i. We have $A\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \cdot \mathbf{u}_1$.

Then $\mathbf{u}_1$ is an eigenvector of A with eigenvalue $\lambda_1 = 1$.
Every non-zero scalar multiple of $\mathbf{u}_1$ is also an eigenvector of A with eigenvalue λ_1.

ii. We have $A\mathbf{u}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = 2\mathbf{u}_2$.

Then $\mathbf{u}_2$ is an eigenvector of A with eigenvalue $\lambda_2 = 2$.
Every non-zero scalar multiple of $\mathbf{u}_2$ is also an eigenvector of A with eigenvalue λ_2.

iii. We have $A\mathbf{u}_3 = \begin{bmatrix} 9 \\ 12 \\ 6 \end{bmatrix} = 3\mathbf{u}_3$.

Then $\mathbf{u}_3$ is an eigenvector of A with eigenvalue $\lambda_3 = 3$.
Every non-zero scalar multiple of $\mathbf{u}_3$ is also an eigenvector of A with eigenvalue λ_3.

(c) Let $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ -5 & 2 & 5 & -1 \end{bmatrix}$, and $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 5 \\ -1 \\ -5 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 1 \\ -5 \\ -3 \\ 15 \end{bmatrix}$.

Note that none of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ is the zero column vector.

i. We have $A\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 1 \cdot \mathbf{u}_1$.

Then $\mathbf{u}_1$ is an eigenvector of A with eigenvalue $\lambda_1 = 1$.

ii. We have $A\mathbf{u}_2 = \begin{bmatrix} -1 \\ -5 \\ 1 \\ 5 \end{bmatrix} = -1 \cdot \mathbf{u}_2$.

Then $\mathbf{u}_2$ is an eigenvector of A with eigenvalue $\lambda_2 = -1$.

iii. We have $A\mathbf{u}_3 = \begin{bmatrix} 3 \\ 3 \\ 9 \\ 9 \end{bmatrix} = 3\mathbf{u}_3$.

Then $\mathbf{u}_3$ is an eigenvector of A with eigenvalue $\lambda_3 = 3$.

iv. We have $A\mathbf{u}_4 = \begin{bmatrix} -3 \\ 15 \\ 9 \\ -45 \end{bmatrix} = -3\mathbf{u}_4$.

Then $\mathbf{u}_4$ is an eigenvector of A with eigenvalue $\lambda_4 = -3$.

v. For each $j = 1, 2, 3, 4$, every non-zero scalar multiple of $\mathbf{u}_j$ is an eigenvector of A with eigenvalue λ_j.

(d) Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, and $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

Note that none of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is the zero column vector.

i. We have $A\mathbf{u}_1 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = 4\mathbf{u}_1$.

Then $\mathbf{u}_1$ is an eigenvector of A with eigenvalue $\lambda_1 = 4$.
Every non-zero scalar multiple of $\mathbf{u}_1$ is also an eigenvector of A with eigenvalue λ_1.

ii. We have $A\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 1 \cdot \mathbf{u}_2$.

Then $\mathbf{u}_2$ is an eigenvector of A with eigenvalue $\lambda_2 = 1$.

iii. We have $A\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 1 \cdot \mathbf{u}_2$.

Then $\mathbf{u}_2$ is an eigenvector of A with eigenvalue $\lambda_3 = 1$.

iv. *Observations.*
Note that $\mathbf{u}_2, \mathbf{u}_3$ are eigenvectors of A with the same eigenvalue, namely, $\lambda_2 = \lambda_3 = 1$.
Then every linear combination of $\mathbf{u}_2, \mathbf{u}_3$ which is not the zero column vector is also an eigenvector of A with eigenvalue λ_2.

v. *Reminder.*
Note that $\mathbf{u}_2, \mathbf{u}_3$ are not non-zero scalar multiples of each other.
As $\mathbf{u}_2, \mathbf{u}_3$ are linearly independent, whenever α, β are non-zero, the linear combination $\alpha\mathbf{u}_2 + \beta\mathbf{u}_3$ is neither a non-zero scalar multiple of $\mathbf{u}_2$ nor a non-zero scalar multiple of $\mathbf{u}_3$.
Nonetheless this does not prevent $\alpha\mathbf{u}_2 + \beta\mathbf{u}_3$ from being an eigenvector of A with eigenvalue λ_2.

6. In order to study questions about eigenvalues and eigenvectors more efficiently we link up the notion of eigenvalues and eigenvectors with what we learnt about homogeneous systems of linear equations.

Lemma (4).

Suppose A is an $(n \times n)$-square matrix, λ is a number, and $\mathbf{v}$ is a non-zero column vector with n entries.

Then the statements $(\star)$, $(\star\star)$ are logically equivalent:—

$(\star)$ $\mathbf{v}$ *is an eigenvector of A with eigenvalue λ.*

$(\star\star)$ $\mathbf{v}$ *is a non-trivial solution of the homogeneous system $\mathcal{LS}(A - \lambda I_n, \mathbf{0}_n)$.*

7. **Proof of Lemma (4).**

Suppose A is an $(n \times n)$-square matrix, λ is a number, and $\mathbf{v}$ is a non-zero column vector with n entries.

(a) Suppose $(\star)$ holds: $\mathbf{v}$ *is an eigenvector of A with eigenvalue λ.*
Then $A\mathbf{v} = \lambda\mathbf{v} = \lambda I_n\mathbf{v}$.
Therefore $(A - \lambda I_n)\mathbf{v} = \mathbf{0}_n$.
Hence $(\star\star)$ holds: $\mathbf{v}$ *is a non-trivial solution of the homogeneous system $\mathcal{LS}(A - \lambda I_n, \mathbf{0}_n)$.*

(b) Suppose $(\star\star)$ holds: $\mathbf{v}$ *is a non-trivial solution of the homogeneous system $\mathcal{LS}(A - \lambda I_n, \mathbf{0}_n)$.*
Then $(A - \lambda I_n)\mathbf{v} = \mathbf{0}_n$.
Therefore $A\mathbf{v} = (A - \lambda I_n + \lambda I_n)\mathbf{v} = (A - \lambda I_n)\mathbf{v} + \lambda I_n\mathbf{v} = \mathbf{0}_n + \lambda\mathbf{v} = \lambda\mathbf{v}$.
Hence $(\star)$ holds: $\mathbf{v}$ *is an eigenvector of A with eigenvalue λ.*

8. Combined with the 'dictionary' on the equivalent formulations for the notion of the invertibility of square matrices, Lemma (4) immediately yields the result below:—

Theorem (5). (Equivalent formulations for the notion of eigenvalues for square matrices.)

Suppose A is an $(n \times n)$-square matrix, and λ is a number. Then the statements below are logically equivalent:—

(1) *λ is an eigenvalue of A.*

(2) *The homogeneous system $\mathcal{LS}(A - \lambda I_n, \mathbf{0}_n)$ has a non-trivial solution.*

(3) *$A - \lambda I_n$ is not invertible.*

(4) *The rank of $A - \lambda I_n$ is at most $n - 1$.*

(5) $\det(A - \lambda I_n) = 0$.

Now suppose one of (1), (2), (3), (4), (5) *holds. (So all of them hold.)*

Further suppose $\mathbf{v}$ *is a column vector with* n *entries.*

Then $\mathbf{v}$ *is an eigenvector of* A *with eigenvalue* λ *if and only if* $\mathbf{v}$ *is a non-trivial solution of* $\mathcal{LS}(A - \lambda I_n, \mathbf{0}_n)$.

Remark. Hence we have yet another small 'extension' of the 'dictionary' about equivalent formulations of invertibility for square matrices:—

Suppose A *is an* $(n \times n)$*-square matrix. Then:—*

(a) A *is not invertible if and only if* 0 *is an eigenvalue of* A.

(b) A *is invertible if and only if* 0 *is not an eigenvalue of* A.

9. When we restrict ourselves to real numbers, and to matrices and vectors with real entries, we can make the connection amongst the notions of eigenvalue, eigenvector, null space and dimension below:—

Theorem (6).

Suppose A *is an* $(n \times n)$*-square matrix with real entries, and* λ *is a real number. Then the statements below are logically equivalent:—*

(I) λ *is an eigenvalue of* A.

(II) $\mathcal{N}(A - \lambda I_n)$ *is of dimension at least* 1 *as a subspace of* $\mathbb{R}^n$ *over the reals.*

Now suppose any one of (I), (II) *holds. (So both of them hold.)*

Further suppose $\mathbf{v} \in \mathbb{R}^n$ *and* $\mathbf{v} \neq \mathbf{0}_n$.

Then $\mathbf{v}$ *is an eigenvector of* A *with eigenvalue* λ *if and only if* $\mathbf{v} \in \mathcal{N}(A - \lambda I_n)$.

Remark on terminology. In the context of Theorem (6), we refer to $\mathcal{N}(A - \lambda I_n)$ as the **eigenspace of** A **with eigenvalue** λ, and we call its dimension as **geometric multiplicity of the eigenvalue** λ **of** A.

Further remark. When we consistently read '$\mathbb{R}$' as '$\mathbb{C}$' and 'real' as 'complex' (and further make sense of phrases like 'null space of a matrix with complex entries' et cetera), we will obtain an analogous result about eigenvalues and eigenvectors for square matrices whose entries are complex numbers.

10. **Example (2). (Example (1) re-done, with help from Theorem (5) and Theorem (6).)**

(a) Let $A = \begin{bmatrix} 13 & 30 \\ -6 & -14 \end{bmatrix}$.

i. For any number λ, we have

$$\det(A - \lambda I_2) = \cdots = \lambda^2 + \lambda - 2 = (\lambda - 1)(\lambda + 2).$$

So the only eigenvalues of A are $1, -2$.
Write $\lambda_1 = 1$, $\lambda_2 = -2$.

ii. Note that $A - \lambda_1 I_2 = \begin{bmatrix} 12 & 30 \\ -6 & -15 \end{bmatrix}$.

By studying the homogeneous system $\mathcal{LS}(A - \lambda_1 I_2, \mathbf{0}_2)$, we find that:—

- $\mathbf{u}_1 = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ is an eigenvector of A with eigenvalue λ_1.
- $\dim(\mathcal{N}(A - \lambda_1 I_2)) = 1$, and $\mathbf{u}_1$ constitutes a basis for $\mathcal{N}(A - \lambda_1 I_2)$.

As a consequence, $\mathbf{v}$ is an eigenvector of A with eigenvalue λ_1 if and only if

- there exists some non-zero number α such that $\mathbf{v} = \alpha \mathbf{u}_1$.

iii. Note that $A - \lambda_2 I_2 = \begin{bmatrix} 15 & 30 \\ -6 & -12 \end{bmatrix}$.

By studying the homogeneous system $\mathcal{LS}(A - \lambda_2 I_2, \mathbf{0}_2)$, we find that:—

- $\mathbf{u}_2 = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$ is an eigenvector of A with eigenvalue λ_2.
- $\dim(\mathcal{N}(A - \lambda_2 I_2)) = 1$, and $\mathbf{u}_2$ constitutes a basis for $\mathcal{N}(A - \lambda_2 I_2)$.

As a consequence, $\mathbf{v}$ is an eigenvector of A with eigenvalue λ_2 if and only if

- there exists some non-zero number α such that $\mathbf{v} = \alpha \mathbf{u}_2$.

(b) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$.

i. For any number λ, we have

$$\det(A - \lambda I_3) = \cdots = -(\lambda - 1)(\lambda - 2)(\lambda - 3).$$

So the only eigenvalues of A are $1, 2, 3$.
Write $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$.

ii. Note that $A - \lambda_1 I_3 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$.

By studying the homogeneous system $\mathcal{LS}(A - \lambda_1 I_3, \mathbf{0}_3)$, we find that:—

- $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector of A with eigenvalue λ_1.
- $\dim(\mathcal{N}(A - \lambda_1 I_3)) = 1$, and $\mathbf{u}_1$ constitutes a basis for $\mathcal{N}(A - \lambda_1 I_3)$.

As a consequence, $\mathbf{v}$ is an eigenvector of A with eigenvalue λ_1 if and only if

- there exists some non-zero number α such that $\mathbf{v} = \alpha \mathbf{u}_1$.

iii. Note that $A - \lambda_2 I_3 = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

By studying the homogeneous system $\mathcal{LS}(A - \lambda_2 I_3, \mathbf{0}_3)$, we find that:—

- $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector of A with eigenvalue λ_2.
- $\dim(\mathcal{N}(A - \lambda_2 I_3)) = 1$, and $\mathbf{u}_2$ constitutes a basis for $\mathcal{N}(A - \lambda_2 I_3)$.

As a consequence, $\mathbf{v}$ is an eigenvector of A with eigenvalue λ_2 if and only if

- there exists some non-zero number α such that $\mathbf{v} = \alpha \mathbf{u}_2$.

iv. Note that $A - \lambda_3 I_3 = \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

By studying the homogeneous system $\mathcal{LS}(A - \lambda_3 I_3, \mathbf{0}_3)$, we find that:—

- $\mathbf{u}_3 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$ is an eigenvector of A with eigenvalue λ_3.
- $\dim(\mathcal{N}(A - \lambda_3 I_3)) = 1$, and $\mathbf{u}_3$ constitutes a basis for $\mathcal{N}(A - \lambda_3 I_3)$.

As a consequence, $\mathbf{v}$ is an eigenvector of A with eigenvalue λ_3 if and only if

- there exists some non-zero number α such that $\mathbf{v} = \alpha \mathbf{u}_3$.

(c) Let $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ -5 & 2 & 5 & -1 \end{bmatrix}$.

i. For any number λ, we have

$$\det(A - \lambda I_4) = \cdots = (\lambda - 1)(\lambda + 1)(\lambda - 3)(\lambda + 3).$$

So the only eigenvalues of A are $1, -1, 3, -3$.
Write $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 3$, $\lambda_4 = -3$.

ii. Note that $A - \lambda_1 I_4 = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ -5 & 2 & 5 & -2 \end{bmatrix}$.

By studying the homogeneous system $\mathcal{LS}(A - \lambda_1 I_4, \mathbf{0}_4)$, we find that:—

- $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ is an eigenvector of A with eigenvalue λ_1.
- $\dim(\mathcal{N}(A - \lambda_1 I_4)) = 1$, and $\mathbf{u}_1$ constitutes a basis for $\mathcal{N}(A - \lambda_1 I_4)$.

As a consequence, $\mathbf{v}$ is an eigenvector of A with eigenvalue λ_1 if and only if

- there exists some non-zero number α such that $\mathbf{v} = \alpha \mathbf{u}_1$.

iii. Note that $A - \lambda_2 I_4 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 1 & 2 & 1 \\ -5 & 2 & 5 & 0 \end{bmatrix}$.

By studying the homogeneous system $\mathcal{LS}(A - \lambda_2 I_4, \mathbf{0}_4)$, we find that:—

- $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 5 \\ -1 \\ -5 \end{bmatrix}$ is an eigenvector of A with eigenvalue λ_2.
- $\dim(\mathcal{N}(A - \lambda_2 I_4)) = 1$, and $\mathbf{u}_2$ constitutes a basis for $\mathcal{N}(A - \lambda_2 I_4)$.

As a consequence, $\mathbf{v}$ is an eigenvector of A with eigenvalue λ_2 if and only if

- there exists some non-zero number α such that $\mathbf{v} = \alpha \mathbf{u}_2$.

iv. Note that $A - \lambda_3 I_4 = \begin{bmatrix} -3 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \\ 2 & 1 & -2 & 1 \\ -5 & 2 & 5 & -4 \end{bmatrix}$.

By studying the homogeneous system $\mathcal{LS}(A - \lambda_3 I_4, \mathbf{0}_4)$, we find that:—

- $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \end{bmatrix}$ is an eigenvector of A with eigenvalue λ_3.
- $\dim(\mathcal{N}(A - \lambda_3 I_4)) = 1$, and $\mathbf{u}_3$ constitutes a basis for $\mathcal{N}(A - \lambda_3 I_4)$.

As a consequence, $\mathbf{v}$ is an eigenvector of A with eigenvalue λ_3 if and only if

- there exists some non-zero number α such that $\mathbf{v} = \alpha \mathbf{u}_3$.

v. Note that $A - \lambda_4 I_4 = \begin{bmatrix} 3 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \\ 2 & 1 & 4 & 1 \\ -5 & 2 & 5 & 2 \end{bmatrix}$.

By studying the homogeneous system $\mathcal{LS}(A - \lambda_4 I_4, \mathbf{0}_4)$, we find that:—

- $\mathbf{u}_4 = \begin{bmatrix} 1 \\ -5 \\ -3 \\ 15 \end{bmatrix}$ is an eigenvector of A with eigenvalue λ_4.
- $\dim(\mathcal{N}(A - \lambda_4 I_4)) = 1$, and $\mathbf{u}_4$ constitutes a basis for $\mathcal{N}(A - \lambda_4 I_4)$.

As a consequence, $\mathbf{v}$ is an eigenvector of A with eigenvalue λ_4 if and only if

- there exists some non-zero number α such that $\mathbf{v} = \alpha \mathbf{u}_4$.

(d) Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

i. For any number λ, we have

$$\det(A - \lambda I_3) = \cdots = -(\lambda - 4)(\lambda - 1)^2.$$

So the only eigenvalues of A are $4, 1$.
Write $\lambda_1 = 4$, $\lambda_2 = 1$.

ii. Note that $A - \lambda_1 I_3 = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$.

By studying the homogeneous system $\mathcal{LS}(A - \lambda_1 I_3, \mathbf{0}_3)$, we find that:—

- $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue λ_1.
- $\dim(\mathcal{N}(A - \lambda_1 I_3)) = 1$, and $\mathbf{u}_1$ constitutes a basis for $\mathcal{N}(A - \lambda_1 I_3)$.

As a consequence, $\mathbf{v}$ is an eigenvector of A with eigenvalue λ_1 if and only if

- there exists some non-zero number α such that $\mathbf{v} = \alpha \mathbf{u}_1$.

iii. Note that $A - \lambda_2 I_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

By studying the homogeneous system $\mathcal{LS}(A - \lambda_2 I_3, \mathbf{0}_3)$, we find that:—

- Each of $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is an eigenvector of A with eigenvalue λ_2.
- $\dim(\mathcal{N}(A - \lambda_2 I_3)) = 2$, and $\mathbf{u}_2, \mathbf{u}_3$ constitute a basis for $\mathcal{N}(A - \lambda_2 I_3)$.

As a consequence, $\mathbf{v}$ is an eigenvector of A with eigenvalue λ_2 if and only if

- there exists some number α_1, α_2, not both zero, such that $\mathbf{v} = \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2$.

11. We now turn to the study of eigenvalues and eigenvectors of some special types of square matrices.

Theorem (7). (Eigenvalues and eigenvectors of diagonal matrices.)

Suppose D be the $(n \times n)$-diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \cdots, \lambda_n$ from left to right. (We do not assume $\lambda_1, \lambda_2, \cdots, \lambda_n$ to be pairwise distinct.)

Then the statements below hold:—

(1) *For each $j = 1, 2, \cdots n$, the number λ_j is an eigenvalue of D with a corresponding eigenvector $\mathbf{e}_j^{(n)}$.*

(2) *$\lambda_1, \lambda_2, \cdots, \lambda_n$ are the only eigenvalues of A.*

(3) *Suppose λ is an eigenvalue of D, and $\lambda_{j_1} = \lambda_{j_2} = \cdots = \lambda_{j_q} = \lambda$ for some $j_1, j_2, \cdots, j_q$ between 1 and n.*

Then every linear combination of $\mathbf{e}_{j_1}^{(n)}, \mathbf{e}_{j_2}^{(n)}, \cdots, \mathbf{e}_{j_q}^{(n)}$ which is not the zero column vector is an eigenvector of D with eigenvalue λ.

In particular:—

(a) *Every non-zero column vector with n entries is an eigenvector of $\mathcal{O}_{n\times n}$ with eigenvalue 0.*

(b) *Every non-zero column vector with n entries is an eigenvector of I_n with eigenvalue 1.*

Proof of Theorem (7). Exercise (about the definitions.)

12. **Theorem (8). (Eigenvalues and 'special eigenvectors' of upper-triangular matrices.)**

Suppose A is an $(n \times n)$-upper-triangular matrix, whose (i, j)-th entry is denoted by a_{ij} for each i, j.

Then the statements below hold:—

(1) *Suppose λ is a number.*

Then λ is an eigenvalue of A if and only if λ is amongst $a_{11}, a_{22}, \cdots, a_{nn}$.

(2) *$\mathbf{e}_1^{(n)}$ is an eigenvector of A, with eigenvalue a_{11}.*

Proof of Theorem (8). Exercise (about the definitions).

Remark. We refrain from stating more about eigenvectors of upper triangular matrices than the statement (2) in the conclusion of Theorem (8). Exactly what other eigenvectors beyond non-zero scalar multiples of $\mathbf{e}_1^{(n)}$ the matrix A may possess depends on what diagonal entries A may have, and how the non-zero entries above its diagonal are 'distributed'.

13. **Example (3). (Eigenvalues and eigenvectors of upper-triangular matrices.)**

(a) Let $A = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$.

The only eigenvalues of A are $1, 2$.

Write $\lambda_1 = 1$, $\lambda_2 = 2$.

i. $\mathbf{u}_1 = \mathbf{e}_1^{(2)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of A with eigenvalue λ_1.

Note that $A - \lambda_1 I_2 = \begin{bmatrix} 0 & 4 \\ 0 & 1 \end{bmatrix}$.

We find that $\dim(\mathcal{N}(A - \lambda_1 I_2)) = 1$, and $\mathbf{u}_1$ constitutes a basis for $\mathcal{N}(A - \lambda_1 I_2)$.

As a consequence, $\mathbf{v}$ is an eigenvector of A with eigenvalue λ_1 if and only if

- there exists some non-zero number α such that $\mathbf{v} = \alpha \mathbf{u}_1$.

ii. Note that $A - \lambda_2 I_2 = \begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix}$.

By studying the homogeneous system $\mathcal{LS}(A - \lambda_2 I_2, \mathbf{0}_2)$, we find that:—

- $\mathbf{u}_2 = \mathbf{e}_2^{(2)} + 4\mathbf{e}_1^{(2)} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue λ_2.
- $\dim(\mathcal{N}(A - \lambda_2 I_2)) = 1$, and $\mathbf{u}_2$ constitutes a basis for $\mathcal{N}(A - \lambda_2 I_2)$.

As a consequence, $\mathbf{v}$ is an eigenvector of A with eigenvalue λ_2 if and only if

- there exists some non-zero number α such that $\mathbf{v} = \alpha \mathbf{u}_2$.

(b) Let $A = \begin{bmatrix} 1 & 4 & 12 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$.

The only eigenvalues of A are $1, 2, 3$.

Write $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$.

i. $\mathbf{u}_1 = \mathbf{e}_1^{(3)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector of A with eigenvalue λ_1.

Note that $A - \lambda_1 I_3 = \begin{bmatrix} 0 & 4 & 12 \\ 0 & 1 & 6 \\ 0 & 0 & 2 \end{bmatrix}$.

We find that $\dim(\mathcal{N}(A - \lambda_1 I_3)) = 1$, and $\mathbf{u}_1$ constitutes a basis for $\mathcal{N}(A - \lambda_1 I_3)$.
As a consequence, $\mathbf{v}$ is an eigenvector of A with eigenvalue λ_1 if and only if

- there exists some non-zero number α such that $\mathbf{v} = \alpha \mathbf{u}_1$.

ii. Note that $A - \lambda_2 I_3 = \begin{bmatrix} -1 & 4 & 12 \\ 0 & 0 & 6 \\ 0 & 0 & 1 \end{bmatrix}$.

By studying the homogeneous system $\mathcal{LS}(A - \lambda_2 I_3, \mathbf{0}_3)$, we find that:—

- $\mathbf{u}_2 = \mathbf{e}_2^{(3)} + 4\mathbf{e}_1^{(3)} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector of A with eigenvalue λ_2.
- $\dim(\mathcal{N}(A - \lambda_2 I_3)) = 1$, and $\mathbf{u}_2$ constitutes a basis for $\mathcal{N}(A - \lambda_2 I_3)$.

As a consequence, $\mathbf{v}$ is an eigenvector of A with eigenvalue λ_2 if and only if

- there exists some non-zero number α such that $\mathbf{v} = \alpha \mathbf{u}_2$.

iii. Note that $A - \lambda_3 I_3 = \begin{bmatrix} -2 & 4 & 12 \\ 0 & -1 & 6 \\ 0 & 0 & 0 \end{bmatrix}$.

By studying the homogeneous system $\mathcal{LS}(A - \lambda_3 I_3, \mathbf{0}_3)$, we find that:—

- $\mathbf{u}_3 = \mathbf{e}_3^{(3)} + 18\mathbf{e}_1^{(3)} + 6\mathbf{e}_2^{(3)} = \begin{bmatrix} 18 \\ 6 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue λ_3.
- $\dim(\mathcal{N}(A - \lambda_3 I_3)) = 1$, and $\mathbf{u}_3$ constitutes a basis for $\mathcal{N}(A - \lambda_3 I_3)$.

As a consequence, $\mathbf{v}$ is an eigenvector of A with eigenvalue λ_3 if and only if

- there exists some non-zero number α such that $\mathbf{v} = \alpha \mathbf{u}_3$.

(c) Let $A = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$.

The only eigenvalue of A is 1.
Write $\lambda_1 = 1$.
$\mathbf{u}_1 = \mathbf{e}_1^{(2)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of A with eigenvalue λ_1.

Note that $A - \lambda_1 I_2 = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}$.

Note that $\dim(\mathcal{N}(A - \lambda_1 I_2)) = 1$.
Then $\mathbf{u}_1$ constitute a basis for $\mathcal{N}(A - \lambda_1 I_2)$.
As a consequence, $\mathbf{v}$ is an eigenvector of A with eigenvalue λ_1 if and only if

- there exists some non-zero number α such that $\mathbf{v} = \alpha \mathbf{u}_1$.

(d) Let $A = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

The only eigenvalue of A is 1.
Write $\lambda_1 = 1$.
$\mathbf{u}_1 = \mathbf{e}_1^{(3)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector of A with eigenvalue λ_1.

Note that $A - \lambda_1 I_3 = \begin{bmatrix} 0 & 4 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

Note that $\dim(\mathcal{N}(A - \lambda_1 I_3)) = 1$.
Then $\mathbf{u}_1$ constitute a basis for $\mathcal{N}(A - \lambda_1 I_3)$.
As a consequence, $\mathbf{v}$ is an eigenvector of A with eigenvalue λ_1 if and only if

- there exists some non-zero number α such that $\mathbf{v} = \alpha \mathbf{u}_1$.

Remark. As Example (3) suggests, when a square matrix A has an eigenvalue λ, all we can immediately say is that there is an eigenvector $\mathbf{v}$ corresponding to the eigenvalue λ and that the non-zero scalar multiple of $\mathbf{v}$ are eigenvectors of A with eigenvalue λ. It is not easy to say anything beyond this, even when A is as 'simple' as an upper-triangular matrix.

14. The example below suggests that even when the entries of a square matrix are all real, it can happen that some of its eigenvalues are non-real, and the corresponding eigenvectors will (have to) involve complex numbers.

Example (4).

(a) Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

i. For any number λ, we have

$$\det(A - \lambda I_2) = \cdots = \lambda^2 - 2\lambda + 2 = (\lambda - 1)^2 + 1 = (\lambda - 1 - i)(\lambda - 1 + i).$$

So the only eigenvalues of A are $1 + i, 1 - i$.
Write $\lambda_1 = 1 + i$, $\lambda_2 = 1 - i$.

ii. Note that $A - \lambda_1 I_2 = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}$.

By studying the homogeneous system $\mathcal{LS}(A - \lambda_1 I_2, \mathbf{0}_2)$, we find that:—

- $\mathbf{u}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue λ_1.

We can further prove that $\mathbf{v}$ is an eigenvector of A with eigenvalue λ_1 if and only if

- there exists some non-zero complex number α such that $\mathbf{v} = \alpha \mathbf{u}_1$.

iii. Note that $A - \lambda_2 I_2 = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}$.

By studying the homogeneous system $\mathcal{LS}(A - \lambda_2 I_2, \mathbf{0}_2)$, we find that:—

- $\mathbf{u}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue λ_2.

We can further prove that $\mathbf{v}$ is an eigenvector of A with eigenvalue λ_2 if and only if

- there exists some non-zero complex number α such that $\mathbf{v} = \alpha \mathbf{u}_2$.

(b) Let $A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

For any number λ, we have

$$\begin{aligned} & \det(A - \lambda I_4) = \cdots \\ = \; & (\lambda - 1)^4 + 1 = [(\lambda - 1)^2 - i][(\lambda - 1)^2 + i] \\ = \; & \left[\lambda - 1 - \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\right]\left[\lambda - 1 - \left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\right]\left[\lambda - 1 - \left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\right]\left[\lambda - 1 - \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\right] \\ = \; & \left[\lambda - \left(1 + \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\right]\left[\lambda - \left(1 - \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\right]\left[\lambda - \left(1 - \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\right]\left[\lambda - \left(1 + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\right]. \end{aligned}$$

So the only eigenvalues of A are

$$\lambda_1 = (1 + \frac{1}{\sqrt{2}}) + \frac{i}{\sqrt{2}}, \quad \lambda_2 = (1 - \frac{1}{\sqrt{2}}) - \frac{i}{\sqrt{2}}, \quad \lambda_3 = (1 - \frac{1}{\sqrt{2}}) + \frac{i}{\sqrt{2}}, \quad \lambda_4 = (1 + \frac{1}{\sqrt{2}}) - \frac{i}{\sqrt{2}}.$$

They are all non-real complex numbers. Each of them corresponds to some eigenvector with non-real entries.

Write $\zeta = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$, $\eta = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$.

The numbers $\bar{\zeta} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$, $\bar{\eta} = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$, are known as the complex conjugates of ζ, η respectively.

i. An eigenvector $\mathbf{u}_1$ of A with eigenvalue λ_1 is given by $\mathbf{u}_1 = \begin{bmatrix} -\bar{\eta} \\ \bar{\eta} \\ -\bar{\eta} \\ 1 \end{bmatrix}$.

ii. An eigenvector $\mathbf{u}_2$ of A with eigenvalue λ_2 is given by $\mathbf{u}_2 = \begin{bmatrix} -\zeta \\ \zeta \\ -\zeta \\ 1 \end{bmatrix}$.

iii. An eigenvector $\mathbf{u}_3$ of A with eigenvalue λ_3 is given by $\mathbf{u}_3 = \begin{bmatrix} -\bar{\zeta} \\ \bar{\zeta} \\ -\bar{\zeta} \\ 1 \end{bmatrix}$.

iv. An eigenvector $\mathbf{u}_4$ of A with eigenvalue λ_4 is given by $\mathbf{u}_4 = \begin{bmatrix} -\eta \\ \eta \\ -\eta \\ 1 \end{bmatrix}$.