5.3.2 Answers to Exercise.

1. (a) 0.
   (b) 0.
   (c) 5040.
   (d) 0.
   (e) 0.
   (f) 0.

2. (a) i. $A = \begin{bmatrix} 2 & 3 & 2 & 2 & 2 \\ 3 & 5 & 3 & 3 & 4 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & x & 1 \end{bmatrix} \xrightarrow{-2R_3+R_1} \xrightarrow{-3R_3+R_2} \xrightarrow{-1R_3+R_4} B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & x & 1 \end{bmatrix}.$

   $B^t = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 & x \\ 0 & 1 & 1 & 4 & 1 \end{bmatrix}.$

   ii. $B^t = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 & x \\ 0 & 1 & 1 & 4 & 1 \end{bmatrix} \xrightarrow{-1R_3+R_5} \xrightarrow{-2R_5+R_2} \xrightarrow{-1R_3+R_2} C^t = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -5 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 & x \\ 0 & 1 & 0 & 2 & 0 \end{bmatrix}.$

   $C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & -5 & 2 & 3 & 2 \\ 0 & 0 & 1 & x & 0 \end{bmatrix}.$

   iii. $\det(A) = 2x - 3$.

   iv. $\det(AB^2A^tC^2) = (2x-3)^6$.

   (b) i. $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$ constitute a basis for $\mathbb{R}^5$ over the reals if and only if $x \neq \frac{3}{2}$.

   ii. $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ constitute a basis for $\mathbb{R}^5$ over the reals if and only if $x \neq \frac{3}{2}$.

3. (a) $\det(U) = (a+1)(b+1)(c+1)(d+1)\left[1 - \frac{a}{a+1} - \frac{b}{b+1} - \frac{c}{c+1} - \frac{d}{d+1}\right].$

   (b) Suppose $a = b = c = d$. (Recall $a \neq -1$.)

   Then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are linearly dependent over the reals if and only if $a = \frac{1}{3}$.

4. Let a be a number, and A be the (5×5)-square matrix given by

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ a & a & a & a & 1 \\ 2 & 2 & 2 & a & 1 \\ 3 & 3 & 2 & a & 1 \\ 4 & 3 & 2 & a & 1 \end{bmatrix}.$$

   (a) Show that $\det(A) = p(a-m)(a-n)$.

   Here m, n, p are some non-zero integers, whose values are independent of that of a, and which satisfies $m < n$. You have to give the values of m, n, p explicitly.

   (b) For which values of a does the homogeneous system of linear equations $\mathcal{LS}(A, \mathbf{0})$ have a non-trivial solution?

(a) $\det(A) = -(a-1)(a-2)$.

(b) $\mathcal{LS}(A, \mathbf{0})$ has a non-trivial solution if and only if $\det(A) = 0$. The latter happens if and only if ($a = 1$ or $a = 2$).

5. (a) ——

   (b) i. a^2.

   ii. $(e_1b_1 + e_2b_2 + e_3b_3)^2$.

6. ——

7. ——

8. (a) $$A \xrightarrow{-1R_1+R_2} \xrightarrow{-1R_1+R_3} \xrightarrow{-1R_1+R_4} \xrightarrow{-1R_1+R_5} \xrightarrow{-1R_1+R_6} B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -a & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-a & 0 & 0 & 0 \\ 0 & 0 & 0 & 2-a & 0 & 0 \\ 0 & 0 & 0 & 0 & 3-a & 0 \\ 0 & 0 & 0 & 0 & 0 & 4-a \end{bmatrix}.$$

   (b) $\det(A) = -a(1-a)(2-a)(3-a)(4-a)$.

   (c) A is invertible if and only if a is not amongst $0, 1, 2, 3, 4$.

9. (a) i. If $a = 0$ or $b = 0$ or $d = 0$ then $\det(A) = 0$.
   Reason: When $a = 0$ or $b = 0$ or $d = 0$, some row of A is a row of 0's.

   ii. If $b = c$ or $b = d$ or $a = c$ then $\det(A) = 0$.
   Reason: When $b = c$ or $b = d$ or $a = c$, two columns of A are scalar multiples of each other.

   (b) i.

$$A = \begin{bmatrix} a & 0 & a & 0 & a \\ b & 0 & c & 0 & b \\ b^2 & 0 & c^2 & 0 & d^2 \\ 0 & ab & 0 & bc & 0 \\ 0 & cd & 0 & ad & 0 \end{bmatrix} \xrightarrow{\frac{1}{a}R_1} \xrightarrow{\frac{1}{b}R_4} \xrightarrow{\frac{1}{d}R_5} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ b & 0 & c & 0 & b \\ b^2 & 0 & c^2 & 0 & d^2 \\ 0 & a & 0 & c & 0 \\ 0 & c & 0 & a & 0 \end{bmatrix}$$

$$\xrightarrow{-cR_1+R_2} \xrightarrow{-c^2R_1+R_3} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ b-c & 0 & 0 & 0 & b-c \\ b^2-c^2 & 0 & 0 & 0 & d^2-c^2 \\ 0 & a & 0 & c & 0 \\ 0 & c & 0 & a & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{b-c}R_2} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ b^2-c^2 & 0 & 0 & 0 & d^2-c^2 \\ 0 & a & 0 & c & 0 \\ 0 & c & 0 & a & 0 \end{bmatrix}$$

$$\xrightarrow{1R_4+R_5} \xrightarrow{\frac{1}{a-c}R_5} \xrightarrow{aR_5+R_4} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ b^2-c^2 & 0 & 0 & 0 & d^2-c^2 \\ 0 & 0 & 0 & a+c & 0 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix}$$

   ii. $$\det(B) = \det(\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ b^2-c^2 & 0 & 0 & 0 & d^2-c^2 \\ 0 & 0 & 0 & a+c & 0 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix}) = (a+c)(d-b)(d+b).$$
   $\det(A) = abd(b-c)(a-c)\det(B) = abd(b-c)(a-c)(a+c)(d-b)(d+b)$.

   (c) ——

10. (a) i. $U \xrightarrow{-1R_1+R_2} \xrightarrow{-1R_1+R_3} \xrightarrow{-1R_1+R_4} \xrightarrow{-1R_1+R_5} \xrightarrow{-1R_1+R_6} V = \begin{bmatrix} 1+a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

ii. $\det(U) = 1 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6$.

(b) i. Yes.

ii. Yes.

iii. Yes.

11. (a) ——

(b) B is invertible if and only if x is not amongst $1, -1, \sqrt{2}, -\sqrt{2}, \sqrt{3}, -\sqrt{3}$.

12. (a) i. ——

ii. $\det(A) = (-1)^{n(n-1)/2} \cdot n^{n-1} \cdot \left(x + \dfrac{n+1}{2}\right)$.

(b) i. $\det(C) = 625b^4(a + 2b)$.

ii. Suppose $b \neq 0$.
The system $\mathcal{LS}(C, \mathbf{d})$ consistent for every column vector $\mathbf{d}$ with five entries if and only if $a \neq -2b$.

13. (a) $x_1 = a$, $x_2 = a^2 - 1$.

(b) ——

(c) i. ——

ii. If $a = 2$ then $x_n = n + 1$ for each positive integer n.
If $a = -2$ then $x_n = (-1)^n(n + 1)$ for each positive integer n.

14. (a) ——

(b) ——

(c) $(\det(A))^2 = 1$ and $(\det(B))^2 = 1$.

(d) $A + B$ is not invertible.

15. ——