5.3.2 Exercise: Determinants.

1. Evaluate the determinants of the square matrices below. (Be observant; try not to start by expanding along a random row/column and computing by brute force.)

(a) $\begin{bmatrix} 1 & 6 & 0 & 4 & 1 & 5 \\ 2 & 5 & 0 & 1 & 6 & 3 \\ 3 & 4 & 0 & 5 & 2 & 1 \\ 4 & 3 & 0 & 2 & 5 & 6 \\ 5 & 2 & 0 & 6 & 3 & 4 \\ 6 & 1 & 0 & 3 & 4 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & 4 & 5 & 6 & 7 \\ 0 & 0 & 0 & 5 & 6 & 7 \\ 0 & 0 & 0 & 0 & 6 & 7 \\ 0 & 0 & 0 & 0 & 0 & 7 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 3 & 1 & 2 & 3 & 4 \\ 2 & 6 & 6 & 7 & 8 & 9 \\ 1 & 3 & 1 & 3 & 5 & 7 \\ 2 & 6 & 3 & 5 & 7 & 9 \\ 1 & 3 & 2 & 4 & 6 & 8 \\ 2 & 6 & 4 & 5 & 6 & 7 \end{bmatrix}$

(b) $\begin{bmatrix} 5 & 7 & 3 & 9 & 1 & 6 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 4 & 7 & 3 & 1 & 6 & 9 \\ 3 & 4 & 8 & 5 & 6 & 1 \\ 1 & 6 & 3 & 5 & 7 & 9 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 & 2 & 2 \\ 1 & 1 & 2 & 2 & 2 & 4 \\ 7 & 6 & 5 & 4 & 3 & 2 \\ 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 7 & 6 & 5 & 4 & 3 \end{bmatrix}$ (f) $\begin{bmatrix} 3 & 5 & 4 & 6 & 7 & 8 \\ 9 & 7 & 5 & 8 & 6 & 4 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 4 & 9 & 6 & 8 & 7 & 5 \\ 2 & 2 & 7 & 2 & 2 & 2 \\ 3 & 3 & 8 & 3 & 3 & 3 \end{bmatrix}$

2. Let x be a number.

   (a) Let

$$A = \begin{bmatrix} 2 & 3 & 2 & 2 & 2 \\ 3 & 5 & 3 & 3 & 4 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & x & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & * & 0 & 0 & 0 \\ 0 & * & 0 & 0 & * \\ 1 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & x & * \end{bmatrix}, \quad C = \begin{bmatrix} 0 & \star & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & \star & \star & 0 \\ 0 & \star & \star & \star & \star \\ 0 & 0 & \star & x & 0 \end{bmatrix}$$

   in which the $*$'s and the $\star$'s stand for various non-zero numbers, whose values are determined according to (I), (II):—

   (I) A is row-equivalent to B under some sequence of three row operations.

   (II) B^t is row-equivalent to C^t under some sequence of three row operations.

   i. Find B, and write down B^t.

   ii. Find C^t, and write down C. (Try to 'retain' the entries of value 0 in B as hard as possible.)

   iii. What is the value of $\det(A)$? Leave your answer in terms of x.

   iv. What is the value of $\det(AB^2A^tC^2)$? Leave your answer in terms of x, (and, if convenient, as the product of some expressions involving x).

   (b) Let

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 3 \\ 5 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 4 \\ x \end{bmatrix}, \mathbf{u}_5 = \begin{bmatrix} 2 \\ 4 \\ 1 \\ 5 \\ 1 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 3 \\ 5 \\ 3 \\ 3 \\ 4 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ x \\ 1 \end{bmatrix}$$

   i. For which values of x are the column vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$ constitute a basis for $\mathbb{R}^5$ over the reals? Justify your answer with reference to your result in the previous part.

   ii. For which values of x do the column vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ constitute a basis for $\mathbb{R}^5$ over the reals? Justify your answer with reference to your result in the previous part.

3. Let a, b, c, d be numbers, none of them being equal to -1.

   Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ be column vectors given by

$$\mathbf{u}_1 = \begin{bmatrix} -1 \\ a \\ a \\ a \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} b \\ -1 \\ b \\ b \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} c \\ c \\ -1 \\ c \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} d \\ d \\ d \\ -1 \end{bmatrix}$$

   and $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4\ ]$.

(a) Verify that

$$\det(U) = m(a+n)(b+n)(c+n)(d+n)\left[p - \frac{a}{a+1} - \frac{b}{b+1} - \frac{c}{c+1} - \frac{d}{d+1}\right].$$

Here m, n, p are some non-zero integers, whose values are independent of that of a, b, c, d. You have to give the values of m, n, p explicitly.

(b) Suppose $a = b = c = d$.

For which value(s) of a are $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ linearly dependent over the reals? Justify your answer.

4. Let a be a number, and A be the (5×5)-square matrix given by

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ a & a & a & a & 1 \\ 2 & 2 & 2 & a & 1 \\ 3 & 3 & 2 & a & 1 \\ 4 & 3 & 2 & a & 1 \end{bmatrix}.$$

(a) Show that $\det(A) = p(a-m)(a-n)$.

Here m, n, p are some non-zero integers, whose values are independent of that of a, and which satisfies $m < n$. You have to give the values of m, n, p explicitly.

(b) For which values of a does the homogeneous system of linear equations $\mathcal{LS}(A, \mathbf{0})$ have a non-trivial solution?

5. (a) Let A be a skew-symmetric $(n \times n)$-matrix.

i. Prove that $\det(A) = (-1)^n \det(A)$.

ii. Hence, or otherwise, prove that if n is odd then A is not invertible.

(b) i. Suppose a is a number. Evaluate $\det(\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix})$.

ii. Suppose $e_1, e_2, e_3, b_1, b_2, b_3$ are numbers.

Show that $\det(\begin{bmatrix} 0 & e_1 & e_2 & e_3 \\ -e_1 & 0 & b_3 & -b_2 \\ -e_2 & -b_3 & 0 & b_1 \\ -e_3 & b_2 & -b_1 & 0 \end{bmatrix}) = (e_1b_1 + pe_2b_2 + qe_3b_3)^m$.

Here m, n, p are some non-zero integers, whose values are independent of that of $e_1, e_2, e_3, b_1, b_2, b_3$. You have to give the values of m, p, q explicitly.

6. In this question the symbols $a, b, c, d, \cdots$ stand for various numbers.

Verify each of the 'identities' below. (Avoid performing 'brute-force' expansion along a row/column until you have reached some determinant with a lot of 0's.)

(a) $\det(\begin{bmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{bmatrix}) = 0$.

(b) $\det(\begin{bmatrix} a & b & a+b \\ b & a+b & a \\ a+b & a & b \end{bmatrix}) = -2(a^3+b^3)$.

(c) $\det(\begin{bmatrix} a & b & c & a+b+c \\ b & c & a+b+c & a \\ c & a+b+c & a & b \\ a+b+c & a & b & c \end{bmatrix}) = 8b(a+b+c)(a^2+c^2)$.

(d) $\det(\begin{bmatrix} a^2 & ab & b^2 \\ b^2 & bc & c^2 \\ c^2 & ca & a^2 \end{bmatrix}) = (a^2-bc)(b^2-ca)(c^2-ab)$.

(e) $\det(\begin{bmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ b+c & c+a & a+b \end{bmatrix}) = -(a-b)(a-c)(b-c)(a+b+c)$.

(f) $\det(\begin{bmatrix} a^2 & bc & c^2+ca \\ a^2+ab & b^2 & ca \\ ab & b^2+bc & c^2 \end{bmatrix}) = 4a^2b^2c^2$.

(g) $\det(\begin{bmatrix} (b+c)^2 & ab & ac \\ ab & (c+a)^2 & bc \\ ac & bc & (a+b)^2 \end{bmatrix}) = 2abc(a+b+c)^3$.

(h) $\det(\begin{bmatrix} 1 & 1 & 1 \\ bc(c-b) & ca(a-c) & ab(b-a) \\ b^2c & c^2a & a^2b \end{bmatrix}) = abc(a^3+b^3+c^3-3abc)$.

(i) $\det(\begin{bmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^3 & b^3 & c^3 & d^3 \\ a^4 & b^4 & c^4 & d^4 \end{bmatrix}) = (b-a)(c-a)(d-a)(c-b)(d-b)(d-c)(ab+ac+ad+bc+bd+cd)$.

(j) $\det(\begin{bmatrix} a & 0 & 0 & 0 & 0 & b \\ 0 & a & 0 & 0 & b & 0 \\ 0 & 0 & a & b & 0 & 0 \\ 0 & 0 & b & a & 0 & 0 \\ 0 & b & 0 & 0 & a & 0 \\ b & 0 & 0 & 0 & 0 & a \end{bmatrix}) = (a^2-b^2)^3$.

7. Let a, b, c, d be numbers. Suppose $a^2+b^2+c^2+d^2=1$.

Show that $\det(\begin{bmatrix} a^2-1 & ab & ac & ad \\ ba & b^2-1 & bc & bd \\ ca & cb & c^2-1 & cd \\ da & db & dc & d^2-1 \end{bmatrix}) = 0$.

8. Let a be a number, and $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1-a & 1 & 1 & 1 & 1 \\ 1 & 1 & 2-a & 1 & 1 & 1 \\ 1 & 1 & 1 & 3-a & 1 & 1 \\ 1 & 1 & 1 & 1 & 4-a & 1 \\ 1 & 1 & 1 & 1 & 1 & 5-a \end{bmatrix}$.

(a) Show that A is row-equivalent to some (6×6)-upper-triangular matrix

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \beta_1 & \alpha & \alpha & \alpha & \alpha \\ 0 & 0 & \beta_2 & \alpha & \alpha & \alpha \\ 0 & 0 & 0 & \beta_3 & \alpha & \alpha \\ 0 & 0 & 0 & 0 & \beta_4 & \alpha \\ 0 & 0 & 0 & 0 & 0 & \beta_5 \end{bmatrix},$$

in which α is some number whose value is independent of that of a, and $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ are some numbers whose values may be dependent on that of a. You are required to give the values of $\alpha, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ explicitly.

(b) Hence, or otherwise, $\det(A)$.

(c) For which values of a is A invertible? Justify your answer.

9. Let a, b, c, d be numbers, and $A = \begin{bmatrix} a & 0 & a & 0 & a \\ b & 0 & c & 0 & b \\ b^2 & 0 & c^2 & 0 & d^2 \\ 0 & ab & 0 & bc & 0 \\ 0 & cd & 0 & ad & 0 \end{bmatrix}$.

(a) i. Suppose $a=0$ or $b=0$ or $d=0$. What is the value of $\det(A)$? Justify your answer.

ii. Suppose $b=c$ or $b=d$ or $a=c$. What is the value of $\det(A)$? Justify your answer.

(b) Suppose $a \neq 0$ and $b \neq 0$ and $d \neq 0$ and $b \neq c$ and $b \neq d$ and $a \neq c$.

i. Show that A is row-equivalent to

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ * & 0 & 0 & 0 & * \\ 0 & 0 & 0 & * & 0 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix}$$

in which the $*$'s stand for various non-zero numbers, whose values may be dependent on a, b, c, d. You must display the various entries of B explicitly.

ii. Evaluate $\det(B)$.
Hence, or otherwise, evaluate $\det(A)$.

(c) Suppose a, b, c, d are all positive real numbers.
Show that A is not invertible if and only if ($a = c$ or $b = c$ or $b = d$).

10. Let $a_1, a_2, a_3, a_4, a_5, a_6$ be numbers, and $U = \left[\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4 \mid \mathbf{u}_5 \mid \mathbf{u}_6\ \right]$, in which

$$\mathbf{u}_1 = \begin{bmatrix} 1+a_1 \\ a_1 \\ a_1 \\ a_1 \\ a_1 \\ a_1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} a_2 \\ 1+a_2 \\ a_2 \\ a_2 \\ a_2 \\ a_2 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} a_3 \\ a_3 \\ 1+a_3 \\ a_3 \\ a_3 \\ a_3 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} a_4 \\ a_4 \\ a_4 \\ 1+a_4 \\ a_4 \\ a_4 \end{bmatrix}, \mathbf{u}_5 = \begin{bmatrix} a_5 \\ a_5 \\ a_5 \\ a_5 \\ 1+a_5 \\ a_5 \end{bmatrix}, \mathbf{u}_6 = \begin{bmatrix} a_6 \\ a_6 \\ a_6 \\ a_6 \\ a_6 \\ 1+a_6 \end{bmatrix}.$$

(a) i. Show that U is row-equivalent to some (6×6)-square matrix

$$V = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 \\ -\alpha & \alpha & 0 & 0 & 0 & 0 \\ -\alpha & 0 & \alpha & 0 & 0 & 0 \\ -\alpha & 0 & 0 & \alpha & 0 & 0 \\ -\alpha & 0 & 0 & 0 & \alpha & 0 \\ -\alpha & 0 & 0 & 0 & 0 & \alpha \end{bmatrix}$$

in which α is some number whose value is independent of the values of $a_1, a_2, a_3, a_4, a_5, a_6$, and $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6$ are some numbers whose values may be dependent of the values of $a_1, a_2, a_3, a_4, a_5, a_6$.
You must give the values of α, $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6$ explicitly.

ii. Hence, or otherwise, evaluate $\det(U)$. Express your answer in terms of $a_1, a_2, a_3, a_4, a_5, a_6$.

(b) Suppose each of $a_1, a_2, a_3, a_4, a_5, a_6$ is a non-negative real number.

i. Is it true that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6$ are linearly independent over the reals? Justify your answer.

ii. Is it true that every vector in $\mathbb{R}^6$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6$ over the reals? Justify your answer.

iii. Do $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6$ constitute a basis for $\mathbb{R}^6$ over the reals? Justify your answer.

11. (a) Let $a_0, a_1, \cdots, a_n, u$ be numbers, and

$$A = \begin{bmatrix} u & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & a_0 \\ -1 & u & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & a_1 \\ 0 & -1 & u & 0 & \ddots & & & 0 & 0 & 0 & a_2 \\ 0 & 0 & -1 & u & \ddots & \ddots & & 0 & 0 & 0 & a_3 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & & \ddots & \ddots & u & 0 & 0 & a_{n-3} \\ 0 & 0 & 0 & 0 & & & \ddots & -1 & u & 0 & a_{n-2} \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & -1 & u & a_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & -1 & a_n \end{bmatrix}.$$

Show that $\det(A) = a_0 + a_1 u + a_2 u^2 + \cdots + a_{n-2} u^{n-2} + a_{n-1} u^{n-1} + a_n u^n$

(b) Let x be a number, and

$$B = \begin{bmatrix} x & 0 & 0 & 0 & 0 & 0 & -6 \\ -1 & x & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & x & 0 & 0 & 0 & 11 \\ 0 & 0 & -1 & x & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & x & 0 & -6 \\ 0 & 0 & 0 & 0 & -1 & x & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

For which value(s) of x is B invertible?

Remark. Take for granted the factorization $y^3 - 6y^2 + 11y - 6 = (y-1)(y-2)(y-3)$ as polynomials.

12. Let n be a positive integer, and x be a number. Define the $(n \times n)$-square matrix A by

$$A = \begin{bmatrix} x+1 & x+2 & x+3 & x+4 & \cdots & \cdots & & x+n-3 & x+n-2 & x+n-1 & x+n \\ x+2 & x+3 & x+4 & x+5 & \cdots & \cdots & & x+n-2 & x+n-1 & x+n & x+1 \\ x+3 & x+4 & x+5 & x+6 & \cdots & & & x+n-1 & x+n & x+1 & x+2 \\ x+4 & x+5 & x+6 & x+7 & \cdots & & \iddots & x+n & x+1 & x+2 & x+3 \\ \vdots & \vdots & \vdots & \vdots & & \iddots & \iddots & \iddots & \iddots & \iddots & \\ \vdots & \vdots & \vdots & & \iddots & \iddots & \iddots & \iddots & \iddots & & \vdots \\ \vdots & \vdots & & \iddots & \iddots & \iddots & \iddots & \iddots & & \vdots & \vdots \\ \vdots & & \iddots & \iddots & \iddots & \iddots & \iddots & & \vdots & \vdots & \vdots \\ & \iddots & \iddots & \iddots & \iddots & \iddots & & \vdots & \vdots & \vdots & \vdots \\ x+n-1 & x+n & x+1 & x+2 & x+3 & \cdots & \cdots & x+n-5 & x+n-4 & x+n-3 & x+n-2 \\ x+n & x+1 & x+2 & x+3 & x+4 & \cdots & \cdots & x+n-4 & x+n-3 & x+n-2 & x+n-1 \end{bmatrix}.$$

(a) i. Show that $\det(A) = \det(B)$, in which the $(n \times n)$-square matrix B is given by

$$B = \left[\begin{array}{c|ccccccccc} x+1 & 1 & 2 & 3 & & \cdots & & n-3 & n-2 & n-1 \\ \hline 1 & 0 & 0 & 0 & & \cdots & & 0 & 0 & -n \\ 1 & 0 & 0 & 0 & & \cdots & & 0 & -n & 0 \\ 1 & 0 & 0 & 0 & & & \iddots & -n & 0 & 0 \\ & & & & & \iddots & \iddots & \iddots & & \\ \vdots & \vdots & \vdots & & \iddots & \iddots & \iddots & & \vdots & \vdots \\ & & & \iddots & \iddots & \iddots & & & & \\ 1 & 0 & 0 & -n & \iddots & & & 0 & 0 & 0 \\ 1 & 0 & -n & 0 & & \cdots & & 0 & 0 & 0 \\ 1 & -n & 0 & 0 & & \cdots & & 0 & 0 & 0 \end{array}\right].$$

ii. Hence, or otherwise, show that $\det(A) = (-1)^{\alpha(n)} \cdot n^{\beta(n)} \cdot \left(x + \dfrac{n+1}{2}\right)$, in which $\alpha(n), \beta(n)$ are some appropriate integers whose values may depend on that of n. You must give the explicit values of $\alpha(n), \beta(n)$.

(b) Let a, b be numbers, and $C = \begin{bmatrix} a & a+b & a+2b & a+3b & a+4b \\ a+b & a+2b & a+3b & a+4b & a \\ a+2b & a+3b & a+4b & a & a+b \\ a+3b & a+4b & a & a+b & a+2b \\ a+4b & a & a+b & a+2b & a+3b \end{bmatrix}$.

i. Evaluate $\det(C)$.

ii. Suppose $b \neq 0$.
For which value(s) of a is the system $\mathcal{LS}(C, \mathbf{d})$ consistent for every column vector $\mathbf{d}$ with five entries?

13. Let a be a number. For each positive integer n, define A_n to be the $(n \times n)$-square matrix given by

$$A_n = \begin{bmatrix} a & 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 0 \\ 1 & a & 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & a & 1 & \ddots & & & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & a & \ddots & \ddots & & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & & \ddots & \ddots & a & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & & & \ddots & 1 & a & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 & a & 1 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 1 & a \end{bmatrix},$$

and define $x_n = \det(A_n)$.

(a) Evaluate x_1, x_2. Express your answers in terms of a.

(b) Show that $x_{n+1} - ax_n + x_{n-1} = 0$ for each integer $n \geq 2$.

(c) i. Suppose $a^2 \neq 4$. Let α, β be the roots of the quadratic polynomial $t^2 - at + 1$. (Note that $\alpha \neq \beta$ because $a^2 \neq 4$.)

Apply mathematical induction to show that $x_n = \dfrac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$ for each positive integer n.

(Hint: Take '$x_n = \dfrac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$ *and* $x_{n+1} = \dfrac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta}$' as the proposition $P(n)$ on which mathematical induction is to be applied.)

ii. Suppose $a^2 = 4$. Compute x_n for each positive integer n.

14. Let A, B be $(n \times n)$-square matrices. Suppose $A^tA = I_n$ and $B^tB = I_n$.

(a) Show that $AA^t = I_n$ and $BB^t = I_n$.

(b) Show that $\det(A)\det(A + B) = \det(I + A^tB) = \det(B)\det(A + B)$.

(c) What are the respective values of $(\det(A))^2, (\det(B))^2$? Justify your answer.

(d) Suppose $\det(AB) < 0$. Is $A + B$ invertible? Justify your answer.

15. (a) Let A be an $(n \times n)$-square matrix, and B be a $(p \times p)$-square matrix.

i. Suppose C is the $((n + p) \times (n + p))$-square matrix given by

$$C = \left[\begin{array}{c|c} A & \mathcal{O}_{n\times p} \\ \hline \mathcal{O}_{p\times n} & B \end{array}\right].$$

A. Suppose B is not invertible. Show that $\det(C) = 0$.

B. Suppose $B = I_p$ (instead). Show that $\det(C) = \det(A)$.

C. (Here we do not assume anything on B.)
Suppose $A = I_n$. Show that $\det(C) = \det(B)$.

D. (We do not assume anything on A or B.)
Hence, or otherwise, show that $\det(C) = \det(A)\det(B)$.

ii. Suppose D is an $(n \times p)$-square matrix, and E is the $((n + p) \times (n + p))$-square matrix given by

$$E = \left[\begin{array}{c|c} A & D \\ \hline \mathcal{O}_{p\times n} & B \end{array}\right].$$

A. Suppose B is invertible. Show that $\det(E) = \det(A)\det(B)$.

B. Hence, or otherwise, show that $\det(E) = \det(A)\det(B)$ whether B is invertible or not.

(b) Let F, G be $(n \times n)$-square matrices. Show that

$$\det(\left[\begin{array}{c|c} F & G \\ \hline G & F \end{array}\right]) = \det(F+G)\det(F-G).$$

(c) Let H, J, K, L be $(n \times n)$-square matrices. Suppose H is invertible.

i. Show that there are some $(n \times n)$-square matrices P, Q, R such that

$$\left[\begin{array}{c|c} P & \mathcal{O}_{n\times n} \\ \hline Q & R \end{array}\right]\left[\begin{array}{c|c} H & J \\ \hline K & L \end{array}\right] = \left[\begin{array}{c|c} I_n & H^{-1}J \\ \hline \mathcal{O}_{n\times n} & L - KH^{-1}J \end{array}\right]$$

ii. Hence, or otherwise, prove the statements below:—

A. If H, K commute with each other then $\det(\left[\begin{array}{c|c} H & J \\ \hline K & L \end{array}\right]) = \det(HL - KJ)$.

B. If H, J commute with each other then $\det(\left[\begin{array}{c|c} H & J \\ \hline K & L \end{array}\right]) = \det(LH - KJ)$.