5.3.1 Appendix: Co-factors, adjoints, and Cramer's Rule.

0. The material in this appendix is supplementary.

1. Recall the notion of *sub-matrices of a square matrix resultant from simultaneous row-and-column deletion.*

   Let A be an $(n \times n)$-square matrix.

   For each $k, \ell = 1, 2, \cdots, n$, the $((n-1) \times (n-1))$-matrix of A resultant from the simultaneous deletion of its k-th row and ℓ-th column is called the (k, ℓ)-**th sub-matrix of** A **(resultant from simultaneous row-and-column deletion)**. *Such a matrix is denoted by $A(k|\ell)$.*

   Definition. (Co-factors of a square matrix.)

   Let A be an $(n \times n)$-square matrix .

   For each $i, j = 1, 2, \cdots, n$, the expression $(-1)^{i+j} \det(A(i|j))$ is called the (i, j)-**th co-factor of** A, *and is denoted by $\mathsf{Cof}_{ij}(A)$.*

2. **Observations on equalities related to cofactors of an arbitrary matrices.**

   Fix any $(n \times n)$-square matrix A, whose (k, ℓ)-th entry is denoted by $a_{k\ell}$ for each k, ℓ.

   (a) Recall what we have learnt about expansion along arbitrary rows and arbitrary columns in A:

   (1) For each $i = 1, 2, \cdots, n$, the equality (i, i) holds:—

   $$\begin{aligned}\det(A) = \; & a_{i1} \cdot (-1)^{i+1} \det(A(i|1)) + a_{i2} \cdot (-1)^{i+2} \det(A(i|2)) + a_{i3} \cdot (-1)^{i+3} \det(A(i|3)) + \cdots\cdots \\ & + a_{i\ell} \cdot (-1)^{i+\ell} \det(A(i|\ell)) + \cdots\cdots + a_{in} \cdot (-1)^{i+n} \det(A(i|n)).\end{aligned}$$

   (2) For each $j = 1, 2, \cdots, n$, the equality (j, j) holds:—

   $$\begin{aligned}\det(A) = \; & a_{1j} \cdot (-1)^{1+j} \det(A(1|j)) + a_{2j} \cdot (-1)^{2+j} \det(A(2|j)) + a_{3j} \cdot (-1)^{3+j} \det(A(3|j)) + \cdots\cdots \\ & + a_{kj} \cdot (-1)^{k+j} \det(A(k|j)) + \cdots\cdots + a_{nj} \cdot (-1)^{n+j} \det(A(n|j)).\end{aligned}$$

   In terms of co-factors, we may re-write (1), (2) as (1'), (2') respectively:—

   (1') For each $i = 1, 2, \cdots, n$, the equality (i, i) holds:—

   $$a_{i1} \cdot \mathsf{Cof}_{i1}(A) + a_{i2} \cdot \mathsf{Cof}_{i2}(A) + a_{i3} \cdot \mathsf{Cof}_{i3}(A) + \cdots + a_{in} \cdot \mathsf{Cof}_{in}(A) = \det(A) \qquad \text{——} \; (i.i)$$

   (2') For each $j = 1, 2, \cdots, n$, the equality $(j.j)$ holds:—

   $$a_{1j} \cdot \mathsf{Cof}_{1j}(A) + a_{2j} \cdot \mathsf{Cof}_{2j}(A) + a_{3j} \cdot \mathsf{Cof}_{3j}(A) + \cdots + a_{nj} \cdot \mathsf{Cof}_{nj}(A) = \det(A) \qquad \text{——} \; (j.j)$$

   (b) Fix any specific i amongst $1, 2, \cdots, n$.

   For each $k = 1, 2, \cdots, n$, whenever $k \neq i$:—

   - the square matrix $A_{R_k \to 1R_i + 0R_k}$, obtained by replacing the k-th row of A with the i-th row of A, has two identical rows at different positions, namely its i-th row and its k-th row,

   and therefore $\det(A_{R_k \to 1R_i + 0R_k}) = 0$.

   Expanding $\det(A_{R_k \to 1R_i + 0R_k})$ along its k-th row, we obtain the equality $(i.k)$:—

   $$a_{i1} \cdot \mathsf{Cof}_{k1}(A) + a_{i2} \cdot \mathsf{Cof}_{k2}(A) + a_{i3} \cdot \mathsf{Cof}_{k3}(A) + \cdots + a_{in} \cdot \mathsf{Cof}_{kn}(A) = 0 \qquad \text{——} \; (i.k)$$

   (c) Fix any specific j amongst $1, 2, \cdots, n$.

   For each $\ell = 1, 2, \cdots, n$, whenever $\ell \neq j$:—

   - the square matrix $A_{C_\ell \to 1C_j + 0C_\ell}$, obtained by replacing the ℓ-th column of A with the j-th column of A, has two identical columns at different positions, namely its j-th column and its ℓ-th column,

   and therefore $\det(A_{C_\ell \to 1C_j + 0C_\ell}) = 0$.

   Expanding $\det(A_{C_\ell \to 1C_j + 0C_\ell})$ along its ℓ-th column, we obtain the equality $(j.\ell)$:—

   $$a_{1j} \cdot \mathsf{Cof}_{1\ell}(A) + a_{2j} \cdot \mathsf{Cof}_{2\ell}(A) + a_{3j} \cdot \mathsf{Cof}_{3\ell}(A) + \cdots + a_{nj} \cdot \mathsf{Cof}_{n\ell}(A) = 0 \qquad \text{——} \; (j.\ell)$$

(d) For each $k = 1, 2, \cdots, n$, we have n simultaneous equalities concerned with column vector

$$\begin{bmatrix} \mathsf{Cof}_{k1}(A) \\ \mathsf{Cof}_{k2}(A) \\ \vdots \\ \mathsf{Cof}_{kn}(A) \end{bmatrix},$$

namely the equalities $(1.k), (2.k), \cdots, ((k-1).k), (k.k), ((k+1).k) \cdots, (n.k)$:—

$$\left\{ \begin{array}{rcrcccrcll} a_{11} \cdot \mathsf{Cof}_{k1}(A) & + & a_{12} \cdot \mathsf{Cof}_{k2}(A) & + & \cdots & + & a_{1n} \cdot \mathsf{Cof}_{kn}(A) & = & 0 & \text{—— } (1.k) \\ a_{21} \cdot \mathsf{Cof}_{k1}(A) & + & a_{22} \cdot \mathsf{Cof}_{k2}(A) & + & \cdots & + & a_{2n} \cdot \mathsf{Cof}_{kn}(A) & = & 0 & \text{—— } (2.k) \\ & & & & & & & \vdots & & \\ a_{k-1,1} \cdot \mathsf{Cof}_{k1}(A) & + & a_{k-1,2} \cdot \mathsf{Cof}_{k2}(A) & + & \cdots & + & a_{k-1,n} \cdot \mathsf{Cof}_{kn}(A) & = & 0 & \text{—— } ((k-1).k) \\ a_{k,1} \cdot \mathsf{Cof}_{k1}(A) & + & a_{k,2} \cdot \mathsf{Cof}_{k2}(A) & + & \cdots & + & a_{k,n} \cdot \mathsf{Cof}_{kn}(A) & = & \det(A) & \text{—— } (k.k) \\ a_{k+1,1} \cdot \mathsf{Cof}_{k1}(A) & + & a_{k+1,2} \cdot \mathsf{Cof}_{k2}(A) & + & \cdots & + & a_{k+1,n} \cdot \mathsf{Cof}_{kn}(A) & = & 0 & \text{—— } ((k+1).k) \\ & & & & & & & \vdots & & \\ a_{n1} \cdot \mathsf{Cof}_{k1}(A) & + & a_{n2} \cdot \mathsf{Cof}_{k2}(A) & + & \cdots & + & a_{nn} \cdot \mathsf{Cof}_{kn}(A) & = & 0 & \text{—— } (n.k) \end{array} \right.$$

Applying the definitions for matrix product and matrix equality, we can re-formulate the above as:—

$$A \begin{bmatrix} \mathsf{Cof}_{k1}(A) \\ \mathsf{Cof}_{k2}(A) \\ \vdots \\ \mathsf{Cof}_{kn}(A) \end{bmatrix} = \det(A)\mathbf{e}_k^{(n)}.$$

(e) Further applying the definition for matrix product, we obtain the equalities

$$A\ \mathsf{Ad}(A) = \left[\ \det(A)\mathbf{e}_1^{(n)} \ \middle|\ \det(A)\mathbf{e}_2^{(n)} \ \middle|\ \cdots \ \middle|\ \det(A)\mathbf{e}_n^{(n)}\ \right] = \det(A)I_n$$

in which $\mathsf{Ad}(A)$ stands for the matrix given by

$$\mathsf{Ad}(A) = \left[\begin{array}{c|c|c|c} \mathsf{Cof}_{11}(A) & \mathsf{Cof}_{21}(A) & \cdots & \mathsf{Cof}_{n1}(A) \\ \mathsf{Cof}_{12}(A) & \mathsf{Cof}_{22}(A) & \cdots & \mathsf{Cof}_{n2}(A) \\ \vdots & \vdots & & \vdots \\ \mathsf{Cof}_{1n}(A) & \mathsf{Cof}_{2n}(A) & \cdots & \mathsf{Cof}_{nn}(A) \end{array} \right]$$

obtained when we placed side-by-side, from left to right, the column vectors

$$\begin{bmatrix} \mathsf{Cof}_{11}(A) \\ \mathsf{Cof}_{12}(A) \\ \vdots \\ \mathsf{Cof}_{1n}(A) \end{bmatrix}, \quad \begin{bmatrix} \mathsf{Cof}_{21}(A) \\ \mathsf{Cof}_{22}(A) \\ \vdots \\ \mathsf{Cof}_{2n}(A) \end{bmatrix}, \quad \cdots, \quad \begin{bmatrix} \mathsf{Cof}_{n1}(A) \\ \mathsf{Cof}_{n2}(A) \\ \vdots \\ \mathsf{Cof}_{nn}(A) \end{bmatrix},$$

to form an $(n \times n)$-square matrix.

The matrix $\mathsf{Ad}(A)$ is called the **adjoint of the square matrix** A.

(f) For each $j = 1, 2, \cdots, n$, we have n simultaneous equalities concerned with column vector

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix},$$

namely the equalities $(j.1), (j.2), \cdots, (j.(j-1)), (j.j), (j.(j+1)) \cdots, (j.n)$:—

$$\left\{ \begin{array}{rcrcccrcll} \mathsf{Cof}_{11}(A) \cdot a_{1j} & + & \mathsf{Cof}_{21}(A) \cdot a_{2j} & + & \cdots & + & \mathsf{Cof}_{n1}(A) \cdot a_{nj} & = & 0 & \text{—— } (j.1) \\ \mathsf{Cof}_{12}(A) \cdot a_{1j} & + & \mathsf{Cof}_{22}(A) \cdot a_{2j} & + & \cdots & + & \mathsf{Cof}_{n2}(A) \cdot a_{nj} & = & 0 & \text{—— } (j.1) \\ & & & & & & & \vdots & & \\ \mathsf{Cof}_{1,j-1}(A) \cdot a_{1j} & + & \mathsf{Cof}_{2,j-1}(A) \cdot a_{2j} & + & \cdots & + & \mathsf{Cof}_{n,j-1}(A) \cdot a_{nj} & = & 0 & \text{—— } (j.(j-1)) \\ \mathsf{Cof}_{1j}(A) \cdot a_{1j} & + & \mathsf{Cof}_{2j}(A) \cdot a_{2j} & + & \cdots & + & \mathsf{Cof}_{nj}(A) \cdot a_{nj} & = & \det(A) & \text{—— } (j.j) \\ \mathsf{Cof}_{1,j+1}(A) \cdot a_{1j} & + & \mathsf{Cof}_{2,j+1}(A) \cdot a_{2j} & + & \cdots & + & \mathsf{Cof}_{n,j+1}(A) \cdot a_{nj} & = & 0 & \text{—— } (j.(j+1)) \\ & & & & & & & \vdots & & \\ \mathsf{Cof}_{1n}(A) \cdot a_{1j} & + & \mathsf{Cof}_{2n}(A) \cdot a_{2j} & + & \cdots & + & \mathsf{Cof}_{nn}(A) \cdot a_{nj} & = & 0 & \text{—— } (j.n) \end{array} \right.$$

Applying the definitions for matrix product and matrix equality, we can re-formulate the above as:—

$$\mathsf{Ad}(A)\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix} = \det(A)\mathbf{e}_j^{(n)}.$$

Further applying the definition for matrix product, we obtain the equalities

$$\mathsf{Ad}(A)\ A = \left[\ \det(A)\mathbf{e}_1^{(n)} \mid \det(A)\mathbf{e}_2^{(n)} \mid \cdots \mid \det(A)\mathbf{e}_n^{(n)}\ \right] = \det(A)I_n$$

3. The above observations are now summarized in the form of a theoretical result:—

Theorem (8). (Product of a matrix with its adjoint.)

Suppose A is an $n \times n$-square matrix.

Then the equalities $A\ \mathsf{Ad}(A) = \det(A)I_n$ and $\mathsf{Ad}(A)\ A = \det(A)I_n$ holds.

Remark on terminology. As stated in the observations above, the $(n \times n)$-square matrix whose (i, j)-th entry is given by the (j, i)-th cofactor $\mathsf{Cof}_{ji}(A)$ of the matrix A is called the **adjoint of the matrix** A.

Written out explicitly in terms of the sub-matrices of A obtained by simultaneous deletions of various rows and columns, $\mathsf{Ad}(A)$ is explicitly given by the equality

$$\mathsf{Ad}(A) = \begin{bmatrix} (-1)^{1+1}\det(A(1|1)) & (-1)^{1+2}\det(A(2|1)) & (-1)^{1+3}\det(A(3|1)) & \cdots & (-1)^{1+n}\det(A(n|1)) \\ (-1)^{2+1}\det(A(1|2)) & (-1)^{2+2}\det(A(2|2)) & (-1)^{2+3}\det(A(3|2)) & \cdots & (-1)^{2+n}\det(A(n|2)) \\ (-1)^{3+1}\det(A(1|3)) & (-1)^{3+2}\det(A(2|3)) & (-1)^{3+3}\det(A(3|3)) & \cdots & (-1)^{3+n}\det(A(n|3)) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-1)^{n+1}\det(A(1|n)) & (-1)^{n+2}\det(A(2|n)) & (-1)^{n+3}\det(A(3|n)) & \cdots & (-1)^{n+n}\det(A(n|n)) \end{bmatrix}$$

4. **Example (6). (Adjoints of square matrices of small sizes.)**

   (a) Suppose $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Then

$$\mathsf{Ad}(A) = \begin{bmatrix} (-1)^{1+1}\det(A(1|1)) & (-1)^{1+2}\det(A(2|1)) \\ (-1)^{2+1}\det(A(1|2)) & (-1)^{1+1}\det(A(2|2)) \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

   (b) Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then

$$\begin{aligned} \mathsf{Ad}(A) &= \begin{bmatrix} (-1)^{1+1}\det(A(1|1)) & (-1)^{1+2}\det(A(2|1)) & (-1)^{1+3}\det(A(3|1)) \\ (-1)^{2+1}\det(A(1|2)) & (-1)^{2+2}\det(A(2|2)) & (-1)^{2+3}\det(A(3|2)) \\ (-1)^{3+1}\det(A(1|3)) & (-1)^{3+2}\det(A(2|3)) & (-1)^{3+3}\det(A(3|3)) \end{bmatrix} \\ &= \begin{bmatrix} a_{22}a_{33} - a_{23}a_{32} & -a_{12}a_{33} + a_{32}a_{13} & a_{12}a_{23} - a_{22}a_{13} \\ -a_{21}a_{33} + a_{31}a_{23} & a_{11}a_{33} - a_{31}a_{13} & -a_{11}a_{23} + a_{21}a_{13} \\ a_{21}a_{32} - a_{31}a_{22} & -a_{11}a_{32} + a_{31}a_{12} & a_{11}a_{22} - a_{21}a_{12} \end{bmatrix} \end{aligned}$$

5. **Theorem (9). (Invertibility of square matrix and its adjoint.)**

Suppose A be an $(n \times n)$-square matrix. Then the statements (1), (2) are logically equivalent:—

   (1) *A is invertible.*

   (2) *$\mathsf{Ad}(A)$ is invertible.*

Moreover, if one of (1), (2) holds (so that both hold), then by $A^{-1} = \dfrac{1}{\det(A)}\mathsf{Ad}(A)$.

Proof of Theorem (9). Exercise. (Make good use of the logical equivalence between the statements '*A is invertible*' and '$\det(A) \neq 0$'.)

6. Now recall the result labelled Theorem ($\star$) below, about systems of linear equations whose coefficient matrices are invertible:—

Theorem ($\star$).

Let A be a $(n \times n)$-square matrix.

Suppose A is invertible.

Then, for any column vector $\mathbf{b}$ with n entries, the system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution, namely $A^{-1}\mathbf{b}$.

7. Theorem (9) combines with Theorem ($\star$) to give a result known as Cramer's Rule, which is one of the earliest discovered result in linear algebra.

Theorem (10). (Cramer's Rule.)

Let A be an $(n \times n)$-square matrix.

Suppose A is invertible.

Then, for any column vector $\mathbf{b}$ with n entries, the system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution, namely $\dfrac{1}{\det(A)}\mathsf{Ad}(A)\mathbf{b}$.

Remark. The essence of this result is in the 'formula' for the matrix inverse A^{-1} of the invertible matrix A that reads:

$$A^{-1} = \frac{1}{\det(A)}\mathsf{Ad}(A).$$

For this reason, we may also refer to this equality as Cramer's Rule.