5.3 Transpose, column operations and determinants.

0. *Assumed background.*
   - What has been covered in Topics 1-3, especially:—
     * 1.3 *Transpose, symmetry and skew-symmetry.*
     * 1.7 *Row operations on matrices.*
   - 5.1 *Determinants of square matrices.*
   - 5.2 *Row operations and determinants.*

   Abstract. We introduce:—

   - the result on the equality between the determinants of a square matrix and its transpose,
   - the 'expansion of determinants along (arbitrary) columns',
   - the multi-linearity and alternating property in columns for determinants, and
   - column operations for the purpose of evaluation of determinants.

   In the *appendix*, we introduce the notions of cofactors and adjoint for a square matrix, and deduce Cramer's Rule.

1. Recall the 'formula' on the 'expansion of determinant of a square matrix along the first row' (through which we formulate the definition for the notion of determinant). It says:—

   Given that A is an $(n \times n)$-square matrix, whose (k, ℓ)-th entry is denoted by $a_{k\ell}$, the equality below holds:—

$$\begin{aligned}\det(A) = \quad & a_{11} \cdot (-1)^{1+1} \det(A(1|1)) + a_{12} \cdot (-1)^{1+2} \det(A(1|2)) + a_{13} \cdot (-1)^{1+3} \det(A(1|3)) + \cdots\cdots \\ & + a_{1j} \cdot (-1)^{1+j} \det(A(1|j)) + \cdots\cdots + a_{1n} \cdot (-1)^{1+n} \det(A(1|n)).\end{aligned}$$

   In fact, it is also fine to 'expand' a determinant 'along the first column' (instead of 'expanding along the first row'), in the sense of the validity of the result below:—

   Theorem (1). ('Expansion of determinant along the first column'.)

   Suppose A is an $(n \times n)$-square matrix, whose (k, ℓ)-th entry is denoted by $a_{k\ell}$.

   Then

$$\begin{aligned}\det(A) = \quad & a_{11} \cdot (-1)^{1+1} \det(A(1|1)) + a_{21} \cdot (-1)^{2+1} \det(A(2|1)) + a_{31} \cdot (-1)^{3+1} \det(A(3|1)) + \cdots\cdots \\ & + a_{i1} \cdot (-1)^{i+1} \det(A(i|1)) + \cdots\cdots + a_{n1} \cdot (-1)^{n+1} \det(A(n|1)).\end{aligned}$$

   Remark on terminology. *This 'formula' is referred to as the 'expansion of* $\det(A)$ *along the first column.*

2. **Proof of Theorem (1).** Omitted. (This argument can be given with the use of mathematical induction, similar to the one about validity in expanding a determinant along arbitrary rows.

   The proposition $P(n)$ on which mathematical induction is applied can be formulated as:—

   For each $(n \times n)$-square matrix A, if the (k, ℓ)-th entry of A is denoted by $a_{k\ell}$ for each k, ℓ, then the equality

$$\begin{aligned}\det(A) = \quad & a_{11} \cdot (-1)^{1+1} \det(A(1|1)) + a_{21} \cdot (-1)^{2+1} \det(A(2|1)) + a_{31} \cdot (-1)^{3+1} \det(A(3|1)) + \cdots\cdots \\ & + a_{i1} \cdot (-1)^{i+1} \det(A(i|1)) + \cdots\cdots + a_{n1} \cdot (-1)^{n+1} \det(A(n|1))\end{aligned}$$

   holds.

   To see how the 'induction step' works out, try to deduce, say, $P(5)$, under the assumption that $P(4)$ is already known to be true. Expand along the first row of the determinant of an arbitrary (5×5)-square matrix, say, C to get a sum involving the determinants of (4×4)-square matrices $C(1|1), C(1|2), C(1|3), C(1|4), C(1|5)$. Then expand along the first columns of the (4×4)-square matrices $C(1|2), C(1|3), C(1|4), C(1|5)$. Now re-group the terms in the resultant sum appropriately, and see what you get.)

3. Using Theorem (1), we can deduce a key result about determinants:—

   Theorem (2). (Equality between respective determinants of a square matrix and of its transpose.)

   Suppose A is a square matrix. Then $\det(A^t) = \det(A)$.

   Proof of Theorem (2).

   (We apply mathematical induction to prove this result.)

   We denote by $P(n)$ the proposition below:—

- *If A is an $(n \times n)$-square matrix, then* $\det(A^t) = \det(A)$.

(a) $P(1)$ follows from definition of determinant immediately.

(b) Let k be a positive integer. Suppose $P(k)$ is true.
(Hence, if B is a $(k \times k)$-square matrix, then $\det(B^t) = \det(B)$.)
Pick any $((k+1) \times (k+1))$-square matrix C, whose (i,j)-th entry is denoted by c_{ij} for each i, j.
Define $D = C^t$, and denote the (i, j)-th entry of D by d_{ij} for each i, j.
Then, for each i, j, we have $d_{ij} = c_{ji}$, and $D(i|j) = (C(j|i))^t$.
Therefore

$$\begin{aligned}
& \det(C^t) = \det(D) \\
= \ & d_{11} \cdot (-1)^{1+1} \det(D(1|1)) + d_{12} \cdot (-1)^{1+2} \det(D(1|2)) + d_{13} \cdot (-1)^{1+3} \det(D(1|3)) + \cdots\cdots \\
& + d_{1k} \cdot (-1)^{1+k} \det(D(1|k)) + \cdots\cdots + d_{1n} \cdot (-1)^{1+n} \det(D(1|n)) \qquad \text{(by expansion along the first row)} \\
= \ & c_{11} \cdot (-1)^{1+1} \det((C(1|1))^t) + c_{21} \cdot (-1)^{1+2} \det((C(2|1))^t) + c_{31} \cdot (-1)^{1+3} \det((C(3|1))^t) + \cdots\cdots \\
& + c_{k1} \cdot (-1)^{1+k} \det((C(k|1))^t) + \cdots\cdots + c_{n1} \cdot (-1)^{1+n} \det((C(n|1))^t) \\
= \ & c_{11} \cdot (-1)^{1+1} \det(C(1|1)) + c_{21} \cdot (-1)^{2+1} \det(C(2|1)) + c_{31} \cdot (-1)^{3+1} \det(C(3|1)) + \cdots\cdots \\
& + c_{k1} \cdot (-1)^{k+1} \det(C(k|1)) + \cdots\cdots + c_{n1} \cdot (-1)^{n+1} \det(C(n|1)) \qquad \text{(by } P(k)\text{)} \\
= \ & \det(C)
\end{aligned}$$

Hence $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for each positive integer n.

4. By virtue of Theorem (2), we obtain out of every theoretical result about determinants another theoretical result by first 'taking transpose' in the matrices and vectors in the former, and then appropriately re-interpretating the latter.
Theorem (3) is an illustration of this point.

Theorem (3). ('Expansion of determinant along arbitrary columns'.)
Suppose A is an $(n \times n)$-square matrix, whose (i, j)-th entry is denoted by a_{ij}.
Then, for each $j = 1, 2, \cdots, n$,

$$\begin{aligned}
\det(A) = \ & a_{1j} \cdot (-1)^{1+j} \det(A(1|j)) + a_{2j} \cdot (-1)^{2+j} \det(A(2|j)) + a_{3j} \cdot (-1)^{3+j} \det(A(3|j)) + \cdots\cdots \\
& + a_{kj} \cdot (-1)^{k+j} \det(A(k|j)) + \cdots\cdots + a_{nj} \cdot (-1)^{n+j} \det(A(n|j)).
\end{aligned}$$

Remark on terminology. The 'formula' in the conclusion is referred to as the 'expansion of $\det(A)$ along the j-th column'.

5. **Example (1). (Explicit display of 'formulae' of 'expansion of determinant along arbitrary columns' for square matrices of small sizes.)**

(a) Suppose $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Then these equalities hold:—

$$\left\{ \begin{array}{rcrcr} \det(A) & = & a_{11} \det(A(1|1)) & - & a_{21} \det(A(2|1)), \\ \det(A) & = & -a_{12} \det(A(1|2)) & + & a_{22} \det(A(2|2)) \end{array} \right.$$

(b) Suppose $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$. Then these equalities below hold:—

$$\left\{ \begin{array}{rcrcrcr} \det(B) & = & b_{11} \det(B(1|1)) & - & b_{21} \det(B(2|1)) & + & b_{31} \det(B(3|1)), \\ \det(B) & = & -b_{12} \det(B(1|2)) & + & b_{22} \det(B(2|2)) & - & b_{32} \det(B(3|2)), \\ \det(B) & = & b_{13} \det(B(1|3)) & - & b_{23} \det(B(2|3)) & + & b_{33} \det(B(3|3)). \end{array} \right.$$

(c) Suppose $C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix}$. Then these equalities hold:—

$$\left\{ \begin{array}{rcrcrcrcr} \det(C) & = & c_{11} \det(C(1|1)) & - & c_{21} \det(C(2|1)) & + & c_{31} \det(C(3|1)) & - & c_{41} \det(C(4|1)), \\ \det(C) & = & -c_{12} \det(C(1|2)) & + & c_{22} \det(C(2|2)) & - & c_{32} \det(C(3|2)) & + & c_{42} \det(C(4|2)), \\ \det(C) & = & c_{13} \det(C(1|3)) & - & c_{23} \det(C(2|3)) & + & c_{33} \det(C(3|3)) & - & c_{43} \det(C(4|3)), \\ \det(C) & = & -c_{14} \det(C(1|4)) & + & c_{24} \det(C(2|4)) & - & c_{34} \det(C(3|4)) & + & c_{44} \det(C(4|4)). \end{array} \right.$$

6. **Proof of Theorem (3).**

   Suppose A is an $(n \times n)$-square matrix, whose (i,j)-th entry is denoted by a_{ij}.

   Write $B = A^t$. For each i, j, denote the (i,j)-th entry of B by b_{ij}.

   By definition, for each i, j, we have $b_{ji} = a_{ij}$, and $B(j|i) = (A(i|j))^t$.

   Then, for each $j = 1, 2, \cdots, n$,

$$\begin{aligned} & \det(A) = \det(A^t) = \det(B) \\ = \ & b_{j1} \cdot (-1)^{j+1} \det(B(j|1)) + b_{j2} \cdot (-1)^{j+2} \det(B(j|2)) + a_{j3} \cdot (-1)^{j+3} \det(B(j|3)) + \cdots\cdots \\ & + b_{jk} \cdot (-1)^{j+k} \det(B(j|k)) + \cdots\cdots + b_{jn} \cdot (-1)^{j+n} \det(B(j|n)) \qquad \text{(by expansion along the } j\text{-th row)} \\ = \ & a_{1j} \cdot (-1)^{j+1} \det((A(1|j))^t) + a_{2j} \cdot (-1)^{j+2} \det((A(2|j))^t) + a_{3j} \cdot (-1)^{j+3} \det((A(3|j))^t) + \cdots\cdots \\ & + a_{kj} \cdot (-1)^{j+k} \det((A(k|j))^t) + \cdots\cdots + a_{nj} \cdot (-1)^{j+n} \det((A(n|j))^t) \\ = \ & a_{1j} \cdot (-1)^{1+j} \det(A(1|j)) + a_{2j} \cdot (-1)^{2+j} \det(A(2|j)) + a_{3j} \cdot (-1)^{3+j} \det(A(3|j)) + \cdots\cdots \\ & + a_{kj} \cdot (-1)^{k+j} \det(A(k|j)) + \cdots\cdots + a_{nj} \cdot (-1)^{n+j} \det(A(n|j)) \end{aligned}$$

7. **Example (2). (Illustration of computation of determinants by 'expansion along columns'.)**

   (a) Suppose $B = \begin{bmatrix} 1 & 7 & 0 \\ 6 & 9 & 8 \\ 0 & 1 & 5 \end{bmatrix}$. Then:—

$$\begin{aligned} \det(B) = \det(\begin{bmatrix} 1 & 7 & 0 \\ 6 & 9 & 8 \\ 0 & 1 & 5 \end{bmatrix}) & = 1 \cdot \det(\begin{bmatrix} & & \\ & 9 & 8 \\ & 1 & 5 \end{bmatrix}) - 6 \cdot \det(\begin{bmatrix} & 7 & 0 \\ & & \\ & 1 & 5 \end{bmatrix}) + 0 \cdot \det(\begin{bmatrix} & 7 & 0 \\ & 9 & 8 \\ & & \end{bmatrix}) \\ & = \cdots = -173. \end{aligned}$$

$$\begin{aligned} \det(B) = \det(\begin{bmatrix} 1 & 7 & 0 \\ 6 & 9 & 8 \\ 0 & 1 & 5 \end{bmatrix}) & = -7 \cdot \det(\begin{bmatrix} & & \\ 6 & & 8 \\ 0 & & 5 \end{bmatrix}) + 9 \cdot \det(\begin{bmatrix} 1 & & 0 \\ & & \\ 0 & & 5 \end{bmatrix}) - 1 \cdot \det(\begin{bmatrix} 1 & & 0 \\ 6 & & 8 \\ & & \end{bmatrix}) \\ & = \cdots = -173. \end{aligned}$$

$$\begin{aligned} \det(B) = \det(\begin{bmatrix} 1 & 7 & 0 \\ 6 & 9 & 8 \\ 0 & 1 & 5 \end{bmatrix}) & = 0 \cdot \det(\begin{bmatrix} & & \\ 6 & 9 & \\ 0 & 1 & \end{bmatrix}) - 8 \cdot \det(\begin{bmatrix} 1 & 7 & \\ & & \\ 0 & 1 & \end{bmatrix}) + 5 \cdot \det(\begin{bmatrix} 1 & 7 & \\ 6 & 9 & \\ & & \end{bmatrix}) \\ & = \cdots = -173. \end{aligned}$$

   (b) Suppose $C = \det(\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix})$. Then:—

$$\begin{aligned} & \det(C) = \det(\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix}) \\ = \ & 1 \cdot \det(\begin{bmatrix} & & & \\ & 5 & 2 & 5 \\ & 9 & 8 & 0 \\ & 9 & 8 & 3 \end{bmatrix}) - 0 \cdot \det(\begin{bmatrix} & 9 & 7 & 7 \\ & & & \\ & 9 & 8 & 0 \\ & 9 & 8 & 3 \end{bmatrix}) + 1 \cdot \det(\begin{bmatrix} & 9 & 7 & 7 \\ & 5 & 2 & 5 \\ & & & \\ & 9 & 8 & 3 \end{bmatrix}) - 1 \cdot \det(\begin{bmatrix} & 9 & 7 & 7 \\ & 5 & 2 & 5 \\ & 9 & 8 & 0 \\ & & & \end{bmatrix}) \\ = \ & \cdots = 15. \end{aligned}$$

$$\begin{aligned} & \det(C) = \det(\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix}) \\ = \ & -9 \cdot \det(\begin{bmatrix} & & & \\ 0 & & 2 & 5 \\ 1 & & 8 & 0 \\ 1 & & 8 & 3 \end{bmatrix}) + 5 \cdot \det(\begin{bmatrix} 1 & & 7 & 7 \\ & & & \\ 1 & & 8 & 0 \\ 1 & & 8 & 3 \end{bmatrix}) - 9 \cdot \det(\begin{bmatrix} 1 & & 7 & 7 \\ 0 & & 2 & 5 \\ & & & \\ 1 & & 8 & 3 \end{bmatrix}) + 9 \cdot \det(\begin{bmatrix} 1 & & 7 & 7 \\ 0 & & 2 & 5 \\ 1 & & 8 & 0 \\ & & & \end{bmatrix}) \\ = \ & \cdots = 15. \end{aligned}$$

$$
\begin{aligned}
\det(C) = \det(&\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix}) \\
= \quad & 7 \cdot \det(\begin{bmatrix} & & \\ 0 & 5 & 5 \\ 1 & 9 & 0 \\ 1 & 9 & 3 \end{bmatrix}) - 2 \cdot \det(\begin{bmatrix} 1 & 9 & 7 \\ & & \\ 1 & 9 & 0 \\ 1 & 9 & 3 \end{bmatrix}) + 8 \cdot \det(\begin{bmatrix} 1 & 9 & 7 \\ 0 & 5 & 5 \\ & & \\ 1 & 9 & 3 \end{bmatrix}) - 8 \cdot \det(\begin{bmatrix} 1 & 9 & 7 \\ 0 & 5 & 5 \\ 1 & 9 & 0 \\ & & \end{bmatrix}) \\
= \quad & \cdots = 15.
\end{aligned}
$$

$$
\begin{aligned}
\det(C) = \det(&\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix}) \\
= \quad & -7 \cdot \det(\begin{bmatrix} & & \\ 0 & 5 & 2 \\ 1 & 9 & 8 \\ 1 & 9 & 8 \end{bmatrix}) + 5 \cdot \det(\begin{bmatrix} 1 & 9 & 7 \\ & & \\ 1 & 9 & 8 \\ 1 & 9 & 8 \end{bmatrix}) - 0 \cdot \det(\begin{bmatrix} 1 & 9 & 7 \\ 0 & 5 & 2 \\ & & \\ 1 & 9 & 8 \end{bmatrix}) + 3 \cdot \det(\begin{bmatrix} 1 & 9 & 7 \\ 0 & 5 & 2 \\ 1 & 9 & 8 \\ & & \end{bmatrix}) \\
= \quad & \cdots = 15.
\end{aligned}
$$

8. **Application of Theorem (2) in evaluation of determinants.**

Before we further expound on the theoretical consequences of Theorem (2), we make demonstrate how Theorem (2) can be used in evaluation of determinants, to supplement the use of row operations.

The 'principle' in how and when Theorem (2) is applied or a row operation is applied is that we want to relate the determinant of a square matrix at every 'intermediate step' of a calculation with:—

- the determinant of a square matrix which contains more 0's in its entries, or
- the determinant of a square matrix of smaller size.

This is illustrated by the example below.

Example (3). (Application of Theorem (2) and row operations in evaluations of determinants.)

(a) *Let* $A = \begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix}$. *We want to evaluate* $\det(A)$.

$$
\begin{aligned}
\det(A) \quad &= \quad \det(\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix}) = \det(\begin{bmatrix} 1 & 0 & 1 & 1 \\ 9 & 5 & 9 & 9 \\ 7 & 2 & 8 & 8 \\ 7 & 5 & 0 & 3 \end{bmatrix}) \quad \begin{pmatrix} \text{Transpose} \\ \text{taken.} \end{pmatrix} \\
&= \quad \det(\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 5 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 7 & 5 & 0 & 3 \end{bmatrix}) \quad \begin{pmatrix} -9R_1 + R_2, \\ -7R_1 + R_3 \\ \text{applied in succession.} \end{pmatrix} \\
&= \quad 5 \cdot \det(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 7 & 0 & 3 \end{bmatrix}) \quad \begin{pmatrix} \text{Expansion along} \\ \text{2-nd row.} \end{pmatrix} \\
&= \quad 5 \cdot \det(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 7 & 0 & 3 \end{bmatrix}) \quad \begin{pmatrix} -1R_2 + R_1 \\ \text{applied.} \end{pmatrix} \\
&= \quad 5 \cdot 1 \cdot \det(\begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}) \quad \begin{pmatrix} \text{Expansion along} \\ \text{1-st row.} \end{pmatrix} \\
&= \quad 5 \cdot 1 \cdot 1 \cdot 3 = 15.
\end{aligned}
$$

(b) *Let* $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 4 & 2 & 2 \\ 1 & 2 & 5 & 2 & 3 \\ 1 & 3 & 3 & 3 & 2 \\ 1 & 4 & 2 & 1 & 4 \end{bmatrix}$. *We want to evaluate* $\det(A)$.

$$\begin{aligned}
\det(A) &= \det(\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 4 & 2 & 2 \\ 1 & 2 & 5 & 2 & 3 \\ 1 & 3 & 3 & 3 & 2 \\ 1 & 4 & 2 & 1 & 4 \end{bmatrix}) = \det(\begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 1 & 2 & 2 & 3 & 4 \\ 1 & 4 & 5 & 3 & 2 \\ 1 & 2 & 2 & 3 & 1 \\ 1 & 2 & 3 & 2 & 4 \end{bmatrix}) \quad \left(\begin{array}{l} \text{Transpose} \\ \text{taken.} \end{array}\right) \\
&= \det(\begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 2 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 1 & 3 \end{bmatrix}) \quad \left(\begin{array}{l} -1R_1 + R_2, \\ -1R_1 + R_3, \\ -1R_1 + R_4, \\ -1R_1 + R_5, \\ \text{applied in succession.} \end{array}\right) \\
&= 1 \cdot \det(\begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 4 & 2 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 3 \end{bmatrix}) \quad \left(\begin{array}{l} \text{Expansion along} \\ \text{1-st column.} \end{array}\right) \\
&= 1 \cdot (-2) \cdot \det(\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 0 \\ 2 & 1 & 3 \end{bmatrix}) \quad \left(\begin{array}{l} \text{Expansion along} \\ \text{1-st column.} \end{array}\right) \\
&= 1 \cdot (-2) \cdot \det(\begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 0 \\ 2 & 1 & 3 \end{bmatrix}) \quad \left(\begin{array}{l} -1R_2 + R_1 \\ \text{applied.} \end{array}\right) \\
&= 1 \cdot (-2) \cdot 3 \cdot \det(\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}) \quad \left(\begin{array}{l} \text{Expansion along} \\ \text{1-st row.} \end{array}\right) \\
&= 1 \cdot (-2) \cdot 3 \cdot (1 \cdot 1 - 2 \cdot 2) = 18.
\end{aligned}$$

9. The manipulations in Example (3) can be further simplified with the use of 'column operations', which are analogous to 'row operations'.

Definition. (Column operations, for matrices of arbitrary sizes.)

Let A be a $(p \times q)$-matrix whose (i, j)-th entry is denoted by a_{ij}, and whose k-th column is denoted by $\mathbf{a}_k$.

(a) *Suppose α is a number.*
When we replace the k-th column $\mathbf{a}_k$ of A by $\alpha\mathbf{a}_i + \mathbf{a}_k$ in which $i \neq k$, to obtain some resultant matrix A', we say we are applying the column operation '$\alpha \cdot C_i + C_k$' to A, and write $A \xrightarrow{\alpha C_i + C_k} A'$.
Such a column operation is called **adding a scalar multiple of one column of A to another column of A.**

(b) *Suppose β is a non-zero number.*
When we replace the k-th column $\mathbf{a}_k$ of A by $\beta\mathbf{a}_k$ to obtain some resultant matrix A', we say we are applying the column operation '$\beta \cdot C_k$' to A, and write $A \xrightarrow{\beta C_k} A'$.
Such a column operation is called **multiplying a non-zero scalar to a column of** A.

(c) *When we interchange the i-th column $\mathbf{a}_i$ and the k-th column $\mathbf{a}_k$ of A, in which $i \neq k$, to obtain some resultant matrix A', we say we are applying the column operation '$C_i \longleftrightarrow C_k$' to A, and write $A \xrightarrow{C_i \leftrightarrow C_k} A'$.*
Such a column operation is called **interchanging two columns of** A.

(d) *We say we are* **applying a column operation** *on A to obtain the $(p \times q)$-matrix A' if and only if A' is the resultant of the application of*
- *one column operation adding a scalar multiple of one column of A to another column of A, or*
- *one column operation multiplying a non-zero scalar to a column of A, or*
- *one column operation interchanging two columns of A.*

10. We can develop a theory concerned with column operations and 'column-operation matrices' analogous to that for row operations and row-operation matrices. For our purpose here, we will be content with formulating the result below which is directly relevant to evaluation of determinants with column operations. The proof of Theorem (4) is no more than an application of Theorem (2) and the definition for column operations (and is left as an exercise).

Theorem (4). (Determinants for a pair of square matrices related through a column operation.)

Let $A, \widehat{A}$ be square matrices of the same size. The statements below hold:—

(a) *Suppose α is a number. Then the statements (1), (1') are logically equivalent:—*

(1) *The application of* $\alpha C_i + C_k$ *on* A *results in* $\widehat{A}$.

(1') *The application of* $\alpha R_i + R_k$ *on* A^t *results in* $\widehat{A}^t$.

Moreover, if any one of (1), (1') *holds (so that both hold), then* $\det(\widehat{A}) = \det(A)$.

(b) *Suppose* β *is a non-zero number. Then the statements* (2), (2')*are logically equivalent:—*

(2) *The application of* βC_k *on* A *results in* $\widehat{A}$.

(2') *The application of* βR_k *on* A^t *results in* $\widehat{A}^t$.

Moreover, if any one of (2), (2') *holds (so that both hold), then* $\det(\widehat{A}) = \beta \det(A)$.

(c) *The statements* (3), (3') *are logically equivalent:—*

(3) *The application of* $C_i \leftrightarrow C_k$ *on* A *results in* $\widehat{A}$.

(3') *The application of* $R_i \leftrightarrow R_k$ *on* A^t *results in* $\widehat{A}^t$.

Moreover, if any one of (3), (3') *holds (so that both hold), then* $\det(\widehat{A}) = -\det(A)$.

11. **Example (4). (Example (3) re-done, with direct reference to column operations.)**

(a) *Let* $A = \begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix}$. *We want to evaluate* $\det(A)$.

$$\begin{aligned}
\det(A) &= \det(\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix}) = \det(\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 3 \end{bmatrix}) \quad \left(\begin{array}{l} -9C_1 + C_2, \\ -7C_1 + C_3 \\ \text{applied in succession.} \end{array}\right) \\
&= 5 \cdot \det(\begin{bmatrix} 1 & 0 & 7 \\ 1 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix}) \quad \left(\begin{array}{l} \text{Expansion along} \\ \text{2-nd column.} \end{array}\right) \\
&= 5 \cdot \det(\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}) \quad \left(\begin{array}{l} -1C_2 + C_1 \\ \text{applied.} \end{array}\right) \\
&= 5 \cdot 1 \cdot \det(\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}) \quad \left(\begin{array}{l} \text{Expansion along} \\ \text{1-st column.} \end{array}\right) \\
&= 5 \cdot 1 \cdot 1 \cdot 3 = 15.
\end{aligned}$$

(b) *Let* $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 4 & 2 & 2 \\ 1 & 2 & 5 & 2 & 3 \\ 1 & 3 & 3 & 3 & 2 \\ 1 & 4 & 2 & 1 & 4 \end{bmatrix}$. *We want to evaluate* $\det(A)$.

$$\begin{aligned}
\det(A) &= \det(\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 4 & 2 & 2 \\ 1 & 2 & 5 & 2 & 3 \\ 1 & 3 & 3 & 3 & 2 \\ 1 & 4 & 2 & 1 & 4 \end{bmatrix}) = \det(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 \\ 1 & 1 & 4 & 1 & 2 \\ 1 & 2 & 2 & 2 & 1 \\ 1 & 3 & 1 & 0 & 3 \end{bmatrix}) \quad \left(\begin{array}{l} -1C_1 + C_2 \\ -1C_1 + C_3 \\ -1C_1 + C_4 \\ -1C_1 + C_5 \\ \text{applied in succession.} \end{array}\right) \\
&= 1 \cdot \det(\begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 4 & 1 & 2 \\ 2 & 2 & 2 & 1 \\ 3 & 1 & 0 & 3 \end{bmatrix}) \quad \left(\begin{array}{l} \text{Expansion along} \\ \text{1-st row.} \end{array}\right) \\
&= 1 \cdot (-2) \cdot \det(\begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \\ 3 & 0 & 3 \end{bmatrix}) \quad \left(\begin{array}{l} \text{Expansion along} \\ \text{1-st row.} \end{array}\right) \\
&= 1 \cdot (-2) \cdot \det(\begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \\ 3 & 0 & 3 \end{bmatrix}) \quad \left(\begin{array}{l} -1C_2 + C_1 \\ \text{applied.} \end{array}\right) \\
&= 1 \cdot (-2) \cdot 3 \cdot \det(\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}) \quad \left(\begin{array}{l} \text{Expansion along} \\ \text{1-st column.} \end{array}\right) \\
&= 1 \cdot (-2) \cdot 3 \cdot (1 \cdot 1 - 2 \cdot 2) = 18.
\end{aligned}$$

12. **Example (5). (Further illustrations on evaluation of determinants through row/column operations.)**

(a) *Let* $A = \begin{bmatrix} 3 & 2 & -1 \\ 4 & 1 & 6 \\ -3 & -1 & 2 \end{bmatrix}$. *We want to evaluate* $\det(A)$.

$$\begin{aligned}
\det(A) &= \det(\begin{bmatrix} 3 & 2 & -1 \\ 4 & 1 & 6 \\ -3 & -1 & 2 \end{bmatrix}) = \det(\begin{bmatrix} 0 & 2 & -1 \\ 9 & 1 & 6 \\ 0 & -1 & 2 \end{bmatrix}) \quad \left(\begin{array}{l} -1C_2 + C_1, \\ 1C_3 + C_1 \\ \text{applied in sucession.} \end{array} \right) \\
&= -9 \cdot \det(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \left(\begin{array}{l} \text{Expansion along} \\ \text{1-st column.} \end{array} \right) \\
&= -9 \cdot [2 \cdot 2 - (-1) \cdot (-1)] = -27
\end{aligned}$$

(b) *Let* $A = \begin{bmatrix} -2 & 3 & 6 & 1 \\ 9 & -2 & -4 & 1 \\ 1 & 3 & 4 & -1 \\ 4 & 1 & 4 & 6 \end{bmatrix}$. *We want to evaluate* $\det(A)$.

$$\begin{aligned}
\det(A) &= \det(\begin{bmatrix} -2 & 3 & 6 & 1 \\ 9 & -2 & -4 & 1 \\ 1 & 3 & 4 & -1 \\ 4 & 1 & 4 & 6 \end{bmatrix}) = \det(\begin{bmatrix} -2 & 3 & 0 & 1 \\ 9 & -2 & 0 & 1 \\ 1 & 3 & -2 & -1 \\ 4 & 1 & 2 & 6 \end{bmatrix}) \quad \left(\begin{array}{l} -2C_2 + C_3 \\ \text{applied.} \end{array} \right) \\
&= \det(\begin{bmatrix} -2 & 3 & 0 & 1 \\ 9 & -2 & 0 & 1 \\ 1 & 3 & -2 & -1 \\ 5 & 4 & 0 & 5 \end{bmatrix}) \quad \left(\begin{array}{l} 1R_3 + R_4 \\ \text{applied.} \end{array} \right) \\
&= -2 \cdot \det(\begin{bmatrix} -2 & 3 & 1 \\ 9 & -2 & 1 \\ 5 & 4 & 5 \end{bmatrix}) \quad \left(\begin{array}{l} \text{Expansion along} \\ \text{3-rd column.} \end{array} \right) \\
&= -2 \cdot \det(\begin{bmatrix} -2 & 3 & 1 \\ 11 & -5 & 0 \\ 15 & -11 & 0 \end{bmatrix}) \quad \left(\begin{array}{l} -1R_1 + R_2 \\ -5R_1 + R_3 \\ \text{applied in succession.} \end{array} \right) \\
&= -2 \cdot \det(\begin{bmatrix} 11 & -5 \\ 15 & -11 \end{bmatrix}) \quad \left(\begin{array}{l} \text{Expansion along} \\ \text{3-rd column.} \end{array} \right) \\
&= -2 \cdot [11 \cdot (-11) - (-5) \cdot 15] = \cdots = 92
\end{aligned}$$

(c) *Let* $A = \begin{bmatrix} 2 & 0 & 2 & 3 \\ 1 & 3 & -1 & 1 \\ -1 & 1 & -1 & 2 \\ 3 & 5 & 4 & 0 \end{bmatrix}$. *We want to evaluate* $\det(A)$.

$$\begin{aligned}
\det(A) &= \det(\begin{bmatrix} 2 & 0 & 2 & 3 \\ 1 & 3 & -1 & 1 \\ -1 & 1 & -1 & 2 \\ 3 & 5 & 4 & 0 \end{bmatrix}) = \det(\begin{bmatrix} 2 & 0 & 0 & 3 \\ 1 & 3 & -2 & 1 \\ -1 & 1 & 0 & 2 \\ 3 & 5 & 1 & 0 \end{bmatrix}) \quad \left(\begin{array}{l} -1C_1 + C_3 \\ \text{applied.} \end{array} \right) \\
&= \det(\begin{bmatrix} 2 & 0 & 0 & 3 \\ 0 & 0 & -2 & -11 \\ -1 & 1 & 0 & 2 \\ 3 & 5 & 1 & 0 \end{bmatrix}) \quad \left(\begin{array}{l} -3R_3 + R_2, \\ -2R_1 + R_2 \\ \text{applied in sucession.} \end{array} \right) \\
&= \det(\begin{bmatrix} 2 & 0 & 0 & 3 \\ 0 & 0 & -2 & -11 \\ 0 & 1 & 0 & 2 \\ 8 & 5 & 1 & 0 \end{bmatrix}) \quad \left(\begin{array}{l} 1C_2 + C_1 \\ \text{applied.} \end{array} \right) \\
&= \det(\begin{bmatrix} 2 & 0 & 0 & 3 \\ 0 & 0 & -2 & -11 \\ 0 & 1 & 0 & 2 \\ 0 & 5 & 1 & -12 \end{bmatrix}) \quad \left(\begin{array}{l} -4R_1 + R_4 \\ \text{applied.} \end{array} \right) \\
&= 2 \cdot \det(\begin{bmatrix} 0 & -2 & -11 \\ 1 & 0 & 2 \\ 5 & 1 & -12 \end{bmatrix}) \quad \left(\begin{array}{l} \text{Expansion along} \\ \text{1-st column.} \end{array} \right) \\
&= 2 \cdot \det(\begin{bmatrix} 0 & -2 & -11 \\ 1 & 0 & 0 \\ 5 & 1 & -22 \end{bmatrix}) \quad \left(\begin{array}{l} -2C_1 + C_3 \\ \text{applied.} \end{array} \right) \\
&= 2 \cdot (-1) \cdot \det(\begin{bmatrix} -2 & -11 \\ 1 & -22 \end{bmatrix}) \quad \left(\begin{array}{l} \text{Expansion along} \\ \text{2-nd column.} \end{array} \right) \\
&= 2 \cdot (-1) \cdot \det(\begin{bmatrix} 0 & -55 \\ 1 & -22 \end{bmatrix}) \quad \left(\begin{array}{l} 2R_2 + R_1 \\ \text{applied.} \end{array} \right) \\
&= 2(-1)[0 \cdot (-22) - (-55) \cdot 1] = -110
\end{aligned}$$

13. At the theoretical level, the validity of the calculations done in Example (4) and Example (5) through column operations is based on the 'multi-linearity in columns' and the 'alternating property in columns' for determinants.

Analogous to the multi-linearity in rows for determinants is the multi-linearity in columns for determinants.

Theorem (5). (Multi-linearity in columns for determinants.)

Let A, B, C be $(n \times n)$-square matrix. For each $i = 1, 2, \cdots, n$, denote the j-th columns by $\mathbf{a}_j, \mathbf{b}_j, \mathbf{c}_j$ respectively.

Let q be an integer between 1 and n. Suppose α, β are numbers. and also suppose:—

(1) $\mathbf{c}_q = \alpha \mathbf{a}_q + \beta \mathbf{b}_q$, *and*

(2) $\mathbf{c}_j = \mathbf{a}_j = \mathbf{b}_j$ *whenever* $j \neq q$.

Then $\det(C) = \alpha \det(A) + \beta \det(B)$.

Remark. In symbolic terms, the conclusion in Theorem (5) reads:—

$$\det([\ C_\sharp \mid \alpha \mathbf{a}_q + \beta \mathbf{b}_q \mid C_\flat\ ]) = \alpha \det([\ C_\sharp \mid \mathbf{a}_q \mid C_\flat\ ]) + \beta \det([\ C_\sharp \mid \mathbf{b}_q \mid C_\flat\ ]),$$

in which:—

- $C_\sharp$ stands for the matrix whose columns are $\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_{q-1}$ from left to right, and
- $C_\flat$ stands for the matrix whose columns are $\mathbf{c}_{q+1}, \cdots, \mathbf{c}_{n-1}, \mathbf{c}_n$ from left to right.

Because of the validity of such an equality concerned with the q-th column of determinants of square matrices, we say that the determinant is **linear in its q-th column**.

Overall, we say the determinant is **multi-linear in its columns**.

14. **Proof of Theorem (5).**

Let A, B, C be $(n \times n)$-square matrix. For each $i = 1, 2, \cdots, n$, denote the j-th columns by $\mathbf{a}_j, \mathbf{b}_j, \mathbf{c}_j$ respectively.

Let q be an integer between 1 and n. Suppose α, β are numbers, and also suppose:—

(1) $\mathbf{c}_q = \alpha \mathbf{a}_q + \beta \mathbf{b}_q$, and

(2) $\mathbf{c}_j = \mathbf{a}_j = \mathbf{b}_j$ whenever $j \neq q$.

Then by assumption, for the square matrices A^t, B^t, C^t, whose respective j-th rows are $\mathbf{a}_j{}^t, \mathbf{b}_j{}^t, \mathbf{c}_j{}^t$ for each j, it happens that:—

(1) $\mathbf{c}_q{}^t = \alpha \mathbf{a}_q{}^t + \beta \mathbf{b}_q{}^t$, and

(2) $\mathbf{c}_j{}^t = \mathbf{a}_j{}^t = \mathbf{b}_j{}^t$ whenever $j \neq q$.

Now, by Theorem (2) and by multi-linearity in rows for determinants, we have

$$\det(C) = \det(C^t) = \alpha \det(A^t) + \beta \det(B^t) = \alpha \det(A) + \beta \det(B).$$

15. Using a similar argument for Theorem (5), we also deduce the alternating property in columns for determinants and deduce that determinants with identical columns in distinct positions are necessarily of value 0. They are Theorem (6) and Theorem (7).

Theorem (6). (Alternating property in columns for determinants.)

Let A, C be $(n \times n)$-square matrices. For each $j = 1, 2, \cdots, n$, denote the j-th columns of A, C by $\mathbf{a}_j, \mathbf{c}_j$ respectively.

Suppose p, q are distinct integers amongst $1, 2, \cdots, n$, and further suppose

(a) $\mathbf{c}_q = \mathbf{a}_p$,

(b) $\mathbf{c}_p = \mathbf{a}_q$, *and*

(c) $\mathbf{c}_j = \mathbf{a}_j$ *whenever* $j \neq p$ *and* $j \neq q$.

Then $\det(C) = -\det(A)$.

Remark. In symbolic terms, the conclusion in Theorem (6) reads:—

$$\det([\ A_\sharp \mid \mathbf{a}_q \mid A_\natural \mid \mathbf{a}_p \mid A_\flat\ ]) = -\det([\ A_\sharp \mid \mathbf{a}_p \mid A_\natural \mid \mathbf{a}_q \mid A_\flat\ ]),$$

in which:—

- $A_\sharp$ stands for the matrix whose columns are $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_{p-1}$ from left to right,
- $A_\natural$ stands for the matrix whose columns are $\mathbf{a}_{p+1}, \mathbf{a}_{p+2}, \cdots, \mathbf{a}_{q-1}$ from left to right,

- $A_\flat$ stands for the matrix whose columns are $\mathbf{a}_{q+2}, \cdots, \mathbf{a}_{n-1}, \mathbf{a}_n$ from left to right.

Because of the validity of such equalities as described in the conclusion of Theorem (6), we say that the determinant is **alternating in its columns**.

16. **Theorem (7). (Determinants with identical columns in distinct positions.)**

    Let A be a square matrix.

    Suppose two columns of A at distinct positions are identical. Then $\det(A) = 0$.

    Remark. Recall the counterpart of Theorem (7) about determinants with identical rows in distinct positions:

    Let B be a square matrix.

    Suppose two rows of B at distinct positions are identical. Then $\det(B) = 0$.

    When carefully re-interpreted, this pair of results, together with the 'formulae' for expansion of determinants along arbitrary rows/columns, will yield 'Cramer's Rule', which is essentially a formula for matrix inverse expressed in terms of determinants. For more detail, refer to the *appendix*.