5.2 Row operations and determinants.

0. *Assumed background.*

   - What has been covered in Topics 1-3, especially:—
     - ∗ 1.7 *Row operations on matrices.*
     - ∗ 1.8 *Row operations and matrix multiplication.*
     - ∗ 3.2 *Invertibility and row operations.*
     - ∗ 3.3 *Various necessary and sufficient conditions for invertibility.*
   - 5.1 *Determinants of square matrices.*

Abstract. We introduce:—

- results on how determinants of row-equivalent square matrices are related to each other, together with an algorithm associated with these results for evaluating determinants of square matrices through row operations,
- results on how the determinant of a product of square matrices is related to the determinants of the individual square matrices forming the product,
- how invertibility can be re-formulated in terms of determinants,
- a formula for the determinant of the matrix inverse of an invertible square matrix (in terms of the determinant of such an invertible square matrix), and
- a formula for the determinants of integral powers of square matrices.

1. We are going to see how the values of the determinants of various square matrices when the square matrices are related through row operations.

   We start by recalling the result on the alternating properties in rows for determinants, which is labelled Theorem $(\star_1)$ below:—

   Theorem $(\star_1)$.

   Let A, C be $(n \times n)$-square matrices. For each $i = 1, 2, \cdots, n$, denote the i-th rows of A, C by $\mathbf{a}_i, \mathbf{c}_i$ respectively.

   Suppose p, q are distinct integers amongst $1, 2, \cdots, n$, and further suppose

   (a) $\mathbf{c}_q = \mathbf{a}_p$,

   (b) $\mathbf{c}_p = \mathbf{a}_q$, *and*

   (c) $\mathbf{c}_i = \mathbf{a}_i$ *whenever* $i \neq p$ *and* $i \neq q$.

   Then $\det(C) = -\det(A)$.

2. The content of Theorem $(\star_1)$ can be immediately re-interpreted in terms of row operations and row-operation matrices.

   Theorem (1).

   Let p, q be distinct integers between 1 and n.

   Suppose A is an $(n \times n)$-square matrix, and $\widetilde{A}$ is resultant from the application on A of the row operation pR_q:

   $$A \xrightarrow{R_p \leftrightarrow R_q} \widetilde{A}$$

   Then $\det(\widetilde{A}) = -\det(A)$.

   In particular, $\det(M[R_p \leftrightarrow R_q]) = -1$.

   Remark. We may also present the equality $\det(\widetilde{A}) = -\det(A)$ equivalently as:—

   $$\det(M[R_p \leftrightarrow R_q]A) = \det(M[R_p \leftrightarrow R_q]) \cdot \det(A).$$

3. Now recall the result on the multi-linearity in rows for determinants, which is labeled Theorem $(\star_2)$ below:—

   Theorem $(\star_2)$. (Multi-linearity in rows in determinants.)

   Let A, B, C be $(n \times n)$-square matrix. For each $i = 1, 2, \cdots, n$, denote the i-th rows by $\mathbf{a}_i, \mathbf{b}_i, \mathbf{c}_i$ respectively.

   Let p be an integer between 1 and n. Suppose α, β are numbers. and also suppose:—

   (1) $\mathbf{c}_p = \beta\mathbf{a}_p + \gamma\mathbf{b}_p$, *and*

(2) $\mathbf{c}_i = \mathbf{a}_i = \mathbf{b}_i$ *whenever* $i \neq p$.

Then $\det(C) = \alpha \det(A) + \beta \det(B)$.

4. A special case in Theorem ($\star_2$) can be re-interpreted in terms row operations and row-operation matrices.

   Theorem (2).

   Let p *be an integer between* 1 *and* n, *and* β *be a non-zero number.*

   Suppose A *is an* $(n \times n)$*-square matrix, and* $\widetilde{A}$ *is resultant from the application on* A *of the row operation* βR_p:

   $$A \xrightarrow{\beta R_p} \widetilde{A}$$

   Then $\det(\widetilde{A}) = \beta \det(A)$.

   In particular, $\det(M[\beta R_p]) = \beta$.

   Remark. The equality '$\det(\widetilde{A}) = \beta \det(A)$' can also be presented equivalently as:—

   $$\det(M[\beta R_p]A) = \det(M[\beta R_p]) \cdot \det(A).$$

   Proof of Theorem (2).

   Let p be an integer between 1 and n, and β be a non-zero number.

   Suppose A is an $(n \times n)$-square matrix, and $\widetilde{A}$ is resultant from the application on A of the row operation βR_p:

   $$A \xrightarrow{\beta R_p} \widetilde{A}.$$

   Denote the i-th row of A by $\mathbf{a}_i$ for each $i = 1, 2, \cdots, n$.

   Denote by $A_\star$ the $(n \times n)$-square matrix obtained from A by replacing its p-th row with ${\mathbf{0}_n}^t$.

   Note that the p-th row of $\widetilde{A}$ is given by $\beta \mathbf{a}_p + 1 \cdot {\mathbf{0}_n}^t$, which is the linear combination of the p-th row of A and the p-th row of $A_\star$ with respect to $\beta, 1$.

   For each i between 1 and n which is not equal to p, the respective i-th rows of $\widetilde{A}, A_\star, A$ are all given by $\mathbf{a}_i$.

   Then by Theorem ($\star_2$), we have $\det(\widetilde{A}) = \beta \det(A) + 1 \det(A_\star)$.

   Since $A_\star$ has an entire row of zeros, we have $\det(A_\star) = 0$.

   Hence $\det(\widetilde{A}) = \beta \det(A)$.

   Note that $M[\beta R_p]$ is resultant from the application on the identity matrix the row operation βR_p.

   Hence $\det(M[\beta R_p]) = \beta$.

5. We have linked up two types of row operations, namely 'interchanging two rows', 'multiplying a row by a non-zero number', with the computation of determinants. We may wonder what can be said of the remaining type of row operations, 'adding a scalar multiple of one row to another'.

   Recall the consequence of Theorem ($\star_1$) below, which is labelled Theorem ($\star_3$) here:

   Theorem ($\star_3$).

   Let A *be a square matrix.*

   Suppose two rows of A *at distinct positions are identical. Then* $\det(A) = 0$.

6. As a consequence of Theorem ($\star_2$) and Theorem ($\star_3$), we have the result below:

   Theorem (3).

   Let p, q *be distinct integers between* 1 *and* n, *and* α *be a number.*

   Suppose A *is an* $(n \times n)$*-square matrix, and* $\widetilde{A}$ *is resultant from the application on* A *of the row operation* $\alpha R_p + R_q$:

   $$A \xrightarrow{\alpha R_p + R_q} \widetilde{A}$$

   Then $\det(\widetilde{A}) = \det(A)$.

   In particular, $\det(M[\alpha R_p + R_q]) = 1$.

   Remark. The equality '$\det(\widetilde{A}) = \det(A)$' can also be presented equivalently as:—

   $$\det(M[\alpha R_p + R_q]A) = \det(M[\alpha R_p + R_q]) \cdot \det(A).$$

7. **Proof of Theorem (3).**

   Let p, q be distinct integers between 1 and n, and α be a number.

   Suppose A is an $(n \times n)$-square matrix, and $\widetilde{A}$ is resultant from the application on A of the row operation $\alpha R_p + R_q$:

$$A \xrightarrow{\alpha R_p + R_q} \widetilde{A}$$

   Denote the i-th row of A by $\mathbf{a}_i$ for each $i = 1, 2, \cdots, n$.

   Denote by $A_\star$ the $(n \times n)$-square matrix obtained from A by replacing its q-th row with $\mathbf{a}_p$.

   Note that the q-th row of $\widetilde{A}$ is given by $\alpha\mathbf{a}_p + 1 \cdot \mathbf{a}_q$, which is the linear combination of the q-th row of $A_\star$ and the q-th row of A with respect to $\alpha, 1$.

   For each i between 1 and n which is not equal to q, the respective i-th rows of $\widetilde{A}, A_\star, A$ are all given by $\mathbf{a}_i$.

   Then by Theorem $(\star_2)$, we have $\det(\widetilde{A}) = \alpha \det(A_\star) + 1 \det(A)$.

   By Theorem $(\star_3)$, since the p-th row and the q-th row of $A_\star$ are the same, we have $\det(A_\star) = 0$.

   Hence $\det(\widetilde{A}) = \det(A)$.

   Note that $M[\alpha R_p + R_q]$ is resultant from the application on the identity matrix of the row operation $\alpha R_p + R_q$.

   Therefore $\det(M[\alpha R_p + R_q]) = 1$.

8. The results above, concerned with row operations, not only serve as useful tools for evaluation of determinants, but also open the way to some highly non-trivial theoretical results about determinants.

   Here we focus on further theoretical discussions on determinants of products of square matrices, and on determinants and invertibility. Combining what we have learnt about row-operation matrices and determinants in Theorem (1), Theorem (2), and Theorem (3), we obtain the result below:—

   Lemma (4).

   Let A, H be square matrices of the same size. Suppose H is a row-operation matrix. Then $\det(HA) = \det(H)\det(A)$.

9. Using Lemma (4), and applying mathematical induction, we deduce the result below:—

   Theorem (5).

   Let $A, H_1, H_2, \cdots, H_k$ be square matrices of the same size. Suppose $H_1, H_2, \cdots, H_k$ are row-operation matrices. Then $\det(H_k \cdots H_2 H_1 A) = \det(H_k) \cdots \det(H_2)\det(H_1)\det(A)$.

10. Recall that every row-operation matrix is invertible. Also recall that every invertible matrix is a product of row-operation matrices. Then Theorem (5) yields the result below immediately:—

    Theorem (6).

    Let B be a square matrix. Suppose B is invertible. Then the statements below hold:—

    (1) $\det(B) \neq 0$.

    (2) *Suppose C is a square matrix, of the same size as B. Then $\det(BC) = \det(B)\det(C)$.*

11. **Proof of Theorem (6).**

    Let B be a square matrix. Suppose B is invertible.

    Then there are some row-operation matrices of the same size as B, say, $H_1, H_2, \cdots, H_k$ so that $B = H_k \cdots H_2 H_1$.

    (1) By Theorem (5), we have $\det(B) = \det(H_k) \cdots \det(H_2)\det(H_1)$.
    By Theorem (1), Theorem (2) and Theorem (3), we have $\det(H_j) \neq 0$ for each $j = 1, 2, \cdots, k$.
    Hence $\det(B) \neq 0$.

    (2) Suppose C is a square matrix, of the same size as B. We have $BC = H_k H_{k-1} \cdots H_2 H_1 C$. Then

$$\begin{aligned} \det(BC) &= \det(H_k H_{k-1} \cdots H_2 H_1 C) \\ &= \det(H_k)\det(H_{k-1}) \cdots \det(H_2)\det(H_1)\det(C) \\ &= \det(B)\det(C). \end{aligned}$$

12. We wonder what we can say about the determinant of a non-invertible square matrix. To answer this question, once again recall that:—

- *A square matrix, say, J, which is a row-echelon form is an upper-triangular matrix, and hence,*
- *such a square matrix J is invertible if and only if J has no non-zero rows.*

Theorem (7).

Let B be a square matrix. Suppose B is not invertible. Then the statements below hold:—

(1) $\det(B) = 0$.

(2) *Suppose C is a square matrix, of the same size as B. Then* $\det(BC) = \det(B)\det(C)$.

13. **Proof of Theorem (7).**

Let B be a square matrix. Suppose B is not invertible.

(1) Pick some row-echelon form $B^\sharp$ which is row-equivalent to B.
Since $B^\sharp$ is row-equivalent to B, there is some invertible matrix G, of the same size as B, so that $B = GB^\sharp$.
Since B is not invertible and G is invertible, $B^\sharp$ is not invertible. Now, since $B^\sharp$ is a non-invertible row-echelon form, $\det(B^\sharp) = 0$.
Then, by Theorem (6), we have $\det(B) = \det(GB^\sharp) = \det(G)\det(B^\sharp) = 0$.

(2) Suppose C is a square matrix, of the same size as B.
We have verified that $\det(B) = 0$. Then $\det(B)\det(C) = 0$.
Note that BC is not invertible; otherwise each of B, C would be invertible.
Modifying the argument above for the statement (1), we also (independently) deduce that $\det(BC) = 0$.
Hence $\det(BC) = 0 = \det(B) = \det(C)$.

14. We re-organize (and enrich) the content of Theorem (6) and Theorem (7) into a pair of results, which are Theorem (8) and Theorem (10) below.

Theorem (8). (Corollary (1) to Theorem (6) and Theorem (7), about determinants of products of pairs of square matrices.)

Suppose B, C are square matrices of the same size. Then $\det(BC) = \det(B)\det(C)$.

Remarks.

(a) A seemingly 'innocent' consequence of Theorem (8) is the validity of the equalities

$$\det(BC) = \det(B)\det(C) = \det(C)\det(B) = \det(CB).$$

It should be remembered, however, that BC, CB are not necessarily equal to each other. This is exactly why Theorem (8) is highly non-trivial.

(b) It should also be remembered that the equality $\det(BC) = \det(B)\det(C)$ 'makes sense' only when B and C are square matrices (of the same size).

15. Using Theorem (8) and applying mathematical induction, we deduce:—

Theorem (9). (Determinants of products of many square matrices.)

Suppose $B_1, B_2, \cdots, B_m$ are square matrices of the same size. Then $\det(B_1B_2\cdots B_m) = \det(B_1)\det(B_2)\cdots\det(B_m)$.

16. **Theorem (10). (Corollary (2) to Theorem (6) and Theorem (7), about a re-formulation of invertibility in terms of determinants.)**

Suppose A be a square matrix. Then the statements (†), (‡) are logically equivalent:—

(†) *A is invertible.*

(‡) $\det(A) \neq 0$.

Moreover, if any one of (†), (‡) holds (so that both hold), then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Remark. We have further 'extended' the 'dictionary' about equivalent formulations for the invertibility of square matrices, this time through the notion of determinants.

The logical equivalence between '*A is invertible*' and '$\det(A) \neq 0$' follows immediately from Theorem (6) and Theorem (7). The only thing that needs to be justified is the equality concerned with $\det(A^{-1})$:—

- Under the assumption that A is invertible, we have $A^{-1}A = I_n$.
  Then $\det(A^{-1})\det(A) = \det(A^{-1}A) = \det(I_n) = 1$.
  As $\det(A) \neq 0$, the number $\dfrac{1}{\det A}$ is well-defined, and the equality $\det(A^{-1}) = \dfrac{1}{\det(A)}$ holds.

17. **Theorem (11). (Determinants of integral powers of square matrices.)**

    Let A be a square matrix. Suppose m is an integer. Then:—

    (a) *If m is positive, then $\det(A^m) = (\det(A))^m$.*

    (b) *If A is invertible, and m is non-positive, then $\det(A^{-m}) = (\det(A))^{-m}$.*

    Remark. Usually we also follow the convention on 'zero-th power for numbers', and declare A^0 to be the identity matrix, even when A is not invertible.

18. **'Algorithm' for evaluating determinants through row operations.**

    Given any arbitrary $(n \times n)$-square matrix, say, A, we may use Theorem (5) (or a re-interpretation of Theorem (5) in terms of row operations), together with what we know about determinants of square matrices which are row-echelon forms, for systematically evaluating $\det(A)$:—

    Step (1).

    Obtain a row-echelon form, say, $A^\sharp$, through some sequence of row operations, say, $\rho_1, \rho_2, \cdots, \rho_k$, starting from A:—

    $$A = A_1 \xrightarrow{\rho_1} A_2 \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_{k-1}} A_{k-1} \xrightarrow{\rho_k} A_k = A^\sharp.$$

    Go to Step (2).

    Step (2).

    Inspect $A^\sharp$ (which is an upper-triangular matrix). Ask:—

    Does $A^\sharp$ have any row of zeros?

    - If *yes*, then conclude $\det(A) = 0$.
    - If *no*, then conclude $\det(A) \neq 0$. To find the exact value of $\det(A)$, go to Step (3).

    Step (3).

    (Here it is supposed that every row of $A^\sharp$ is a non-zero row.)

    - Identify all the first non-zero entries of $A^\sharp$, say, $\alpha_1, \alpha_2, \cdots, \alpha_n$, from top to bottom.
    - Identify those row operations amongst $\rho_1, \rho_2, \cdots, \rho_{k-1}, \rho_k$ which are of the type 'interchanging two rows'. Suppose there are p such row operations.
    - Identify those row operations amongst $\rho_1, \rho_2, \cdots, \rho_{k-1}, \rho_k$ which are of the type 'multiplying a row by a non-zero number', and identify those non-zero numbers used.
      Suppose there are q such row operations, and those non-zero numbers used are $\beta_{j_1}, \beta_{j_2}, \cdots, \beta_{j_q}$.

    Then conclude $\det(A^\sharp) = \alpha_1\alpha_2\cdots\alpha_n$, and furthermore, $\det(A) = \dfrac{(-1)^p \cdot \alpha_1\alpha_2\cdots\alpha_n}{\beta_{j_1}\beta_{j_2}\cdots\beta_{j_q}}$.

19. **Example (1). (Illustration of the 'algorithm' for evaluating determinants through row operations.)**

    (a) *Let* $A = \begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix}$. *We want to evaluate* $\det(A)$.

    Obtain from A a row-echelon form $A^\sharp$ which is row-equivalent to A:

    $$A = \begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix} \xrightarrow{-1R_1+R_3} \xrightarrow{-1R_1+R_4} \begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 1 & -4 \end{bmatrix} \xrightarrow{-1R_3+R_4} \begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 3 \end{bmatrix} = A^\sharp$$

    We have $\det(A^\sharp) = 15$. Inspecting the row operations involved, we further conclude that $\det(A) = 15$.

(b) *Let* $A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 1 \\ 3 & 5 & 4 \end{bmatrix}$. *We want to evaluate* $\det(A)$.

Obtain from A a row-echelon form $A^\sharp$ which is row-equivalent to A:

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 1 \\ 3 & 5 & 4 \end{bmatrix} \xrightarrow{-1R_1+R_2} \xrightarrow{-3R_1+R_3} \begin{bmatrix} 1 & 3 & 5 \\ 0 & -2 & -4 \\ 0 & -4 & -11 \end{bmatrix} \xrightarrow{-2R_2+R_3} \begin{bmatrix} 1 & 3 & 5 \\ 0 & -2 & -4 \\ 0 & 0 & -3 \end{bmatrix} = A^\sharp$$

We have $\det(A^\sharp) = 6$. Inspecting the row operations involved, we further conclude that $\det(A) = 6$.

(c) *Let* $A = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & 5 & 7 & 15 \\ 3 & 8 & 10 & 20 \\ 4 & 13 & 17 & 38 \end{bmatrix}$. *We want to evaluate* $\det(A)$.

Obtain from A a row-echelon form $A^\sharp$ which is row-equivalent to A:

$$A = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & 5 & 7 & 15 \\ 3 & 8 & 10 & 20 \\ 4 & 13 & 17 & 38 \end{bmatrix} \xrightarrow{-2R_1+R_2} \xrightarrow{-3R_1+R_3} \xrightarrow{-4R_1+R_4} \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 1 & 2 \\ 0 & 5 & 5 & 14 \end{bmatrix}$$

$$\xrightarrow{-2R_2+R_3} \xrightarrow{-5R_2+R_4} \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & -1 \end{bmatrix} = A^\sharp$$

We have $\det(A^\sharp) = 1$. Inspecting the row operations involved, we further conclude that $\det(A) = 1$.

(d) *Let* $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 8 & 14 \\ -1 & 3 & 7 \end{bmatrix}$. *We want to evaluate* $\det(A)$.

Obtain from A a row-echelon form $A^\sharp$ which is row-equivalent to A:

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 8 & 14 \\ -1 & 3 & 7 \end{bmatrix} \xrightarrow{-2R_1+R_2} \xrightarrow{1R_1+R_3} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 4 \\ 0 & 6 & 12 \end{bmatrix} \xrightarrow{-3R_2+R_3} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix} = A^\sharp$$

We have $\det(A^\sharp) = 0$. Hence $\det(A) = 0$.

(e) *Let* $A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 4 & 14 & 24 & 36 \\ 1 & 9 & 17 & 28 \\ 2 & 4 & 6 & 12 \end{bmatrix}$. *We want to evaluate* $\det(A)$.

Obtain from A a row-echelon form $A^\sharp$ which is row-equivalent to A:

$$A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 4 & 14 & 24 & 36 \\ 1 & 9 & 17 & 28 \\ 2 & 4 & 6 & 12 \end{bmatrix} \xrightarrow{-4R_1+R_2} \xrightarrow{-1R_1+R_3} \xrightarrow{-2R_1+R_4} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 2 & 4 & 8 \\ 0 & 6 & 12 & 21 \\ 0 & -2 & -4 & -2 \end{bmatrix}$$

$$\xrightarrow{-3R_2+R_3} \xrightarrow{1R_2+R_4} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 6 \end{bmatrix} \xrightarrow{2R_3+R_4} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A^\sharp$$

We have $\det(A^\sharp) = 0$. Hence $\det(A) = 0$.

20. In practice, we tend to incorporate the use of the following result into the 'algorithm' for evaluating determinants through row operations:—

- *If C is a square matrix in which a row/column has at most one non-zero entry which is located at, say, its (k, ℓ)-th entry, whose value is γ, then* $\det(C) = \gamma \cdot (-1)^{k+\ell} \det(C(k|\ell))$.

We also tend to 'merge' the sequence of row operations into a chain of equalities about determinants involved in this sequence of row operations.

The point is to reduce the amount of writing needed for the presentation of the calculations.

Example (2). (Example (1) re-done.)

(a) *Let* $A = \begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix}$. *We want to evaluate* $\det(A)$.

$$\begin{aligned}
\det(A) &= \det(\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix}) = \det(\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 1 & -4 \end{bmatrix}) \quad \begin{pmatrix} -1R_1 + R_3, \\ -1R_1 + R_4 \\ \text{applied in succession.} \end{pmatrix} \\
&= \det(\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 3 \end{bmatrix}) \quad (-1R_3 + R_4, \text{applied. }) \\
&= 1 \cdot 5 \cdot 1 \cdot 3 = 15
\end{aligned}$$

(b) *Let* $A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 1 \\ 3 & 5 & 4 \end{bmatrix}$. *We want to evaluate* $\det(A)$.

$$\begin{aligned}
\det(A) &= \det(\begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 1 \\ 3 & 5 & 4 \end{bmatrix}) = \det(\begin{bmatrix} 1 & 3 & 5 \\ 0 & -2 & -4 \\ 0 & -4 & -11 \end{bmatrix}) \quad \begin{pmatrix} -1R_1 + R_2, \\ -3R_1 + R_3 \\ \text{applied in succession.} \end{pmatrix} \\
&= 1 \cdot \det(\begin{bmatrix} -2 & -4 \\ -4 & -11 \end{bmatrix}) \quad \begin{pmatrix} \text{Expansion along} \\ \text{1-st row.} \end{pmatrix} \\
&= 1 \cdot \det(\begin{bmatrix} -2 & -4 \\ 0 & -3 \end{bmatrix}) \quad \begin{pmatrix} -2R_1 + R_2 \\ \text{applied.} \end{pmatrix} \\
&= 1 \cdot (-2) \cdot (-3) = 6
\end{aligned}$$

(c) *Let* $A = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & 5 & 7 & 15 \\ 3 & 8 & 10 & 20 \\ 4 & 13 & 17 & 38 \end{bmatrix}$. *We want to evaluate* $\det(A)$.

$$\begin{aligned}
\det(A) &= \det(\begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & 5 & 7 & 15 \\ 3 & 8 & 10 & 20 \\ 4 & 13 & 17 & 38 \end{bmatrix}) = \det(\begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 1 & 2 \\ 0 & 5 & 5 & 14 \end{bmatrix}) \quad \begin{pmatrix} -2R_1 + R_2, \\ -3R_1 + R_3, \\ -4R_1 + R_4 \\ \text{applied in succession.} \end{pmatrix} \\
&= 1 \cdot \det(\begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 2 \\ 5 & 5 & 14 \end{bmatrix}) \quad \begin{pmatrix} \text{Expansion along} \\ \text{1-st row.} \end{pmatrix} \\
&= 1 \cdot \det(\begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & 0 & -1 \end{bmatrix}) \quad \begin{pmatrix} -2R_1 + R_2, \\ -5R_1 + R_3 \\ \text{applied in succession.} \end{pmatrix} \\
&= 1 \cdot 1 \cdot (-1) \cdot (-1) = 1
\end{aligned}$$

(d) *Let* $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 8 & 14 \\ -1 & 3 & 7 \end{bmatrix}$. *We want to evaluate* $\det(A)$.

$$\begin{aligned}
\det(A) &= \det(\begin{bmatrix} 1 & 3 & 5 \\ 2 & 8 & 14 \\ -1 & 3 & 7 \end{bmatrix}) = \det(\begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 4 \\ 0 & 6 & 12 \end{bmatrix}) \quad \begin{pmatrix} -2R_1 + R_2, \\ 1R_1 + R_3 \\ \text{applied in succession.} \end{pmatrix} \\
&= 1 \cdot \det(\begin{bmatrix} 2 & 4 \\ 6 & 12 \end{bmatrix}) \quad \begin{pmatrix} \text{Expansion along} \\ \text{1-st row.} \end{pmatrix} \\
&= 1 \cdot \det(\begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}) \quad \begin{pmatrix} -3R_1 + R_2 \\ \text{applied.} \end{pmatrix} \\
&= 0
\end{aligned}$$

(e) *Let* $A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 4 & 14 & 24 & 36 \\ 1 & 9 & 17 & 28 \\ 2 & 4 & 6 & 12 \end{bmatrix}$. *We want to evaluate* $\det(A)$.

$$\begin{aligned}
\det(A) &= \det\left(\begin{bmatrix} 1 & 3 & 5 & 7 \\ 4 & 14 & 24 & 36 \\ 1 & 9 & 17 & 28 \\ 2 & 4 & 6 & 12 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 2 & 4 & 8 \\ 0 & 6 & 12 & 21 \\ 0 & -2 & -4 & -2 \end{bmatrix}\right) \quad \left(\begin{array}{l} -4R_1 + R_2, \\ -1R_1 + R_3, \\ -2R_1 + R_4 \\ \text{applied in succession.} \end{array}\right) \\
&= 1 \cdot \det\left(\begin{bmatrix} 2 & 4 & 8 \\ 6 & 12 & 21 \\ -2 & -4 & -2 \end{bmatrix}\right) \quad \left(\begin{array}{l} \text{Expansion along} \\ \text{1-st row.} \end{array}\right) \\
&= 1 \cdot \det\left(\begin{bmatrix} 2 & 4 & 8 \\ 0 & 0 & -3 \\ 0 & 0 & 6 \end{bmatrix}\right) \quad \left(\begin{array}{l} -3R_2 + R_3, \\ 1R_2 + R_4 \\ \text{applied in succession.} \end{array}\right) \\
&= 1 \cdot 2 \cdot 0 \cdot 6 = 0
\end{aligned}$$