

5.1.1 Appendix: Expansion of determinants along arbitrary rows.

0. The material in this appendix is supplementary.

1. We are going to illustrate the key idea in the ‘inductive step’ in mathematical induction argument for Theorem (1):

Theorem (1). (‘Expansion of determinant along arbitrary rows’.)

Suppose A is an $(n \times n)$ -square matrix, whose (i, j) -th entry is denoted by a_{ij} .

Then, for each $i = 1, 2, \dots, n$,

$$\det(A) = \begin{array}{l} a_{i1} \cdot (-1)^{i+1} \det(A(i|1)) + a_{i2} \cdot (-1)^{i+2} \det(A(i|2)) + a_{i3} \cdot (-1)^{i+3} \det(A(i|3)) + \dots \\ + a_{i\ell} \cdot (-1)^{i+\ell} \det(A(i|\ell)) + \dots + a_{in} \cdot (-1)^{i+n} \det(A(i|n)). \end{array}$$

2. The proposition $P(n)$ under which we apply mathematical induction is given by:—

For each $(n \times n)$ -square matrix A , if the (k, ℓ) -th entry of A is denoted by $a_{k\ell}$ for each k, ℓ , then, for each $i = 1, 2, \dots, n$, the equality

$$\det(A) = \begin{array}{l} a_{i1} \cdot (-1)^{i+1} \det(A(i|1)) + a_{i2} \cdot (-1)^{i+2} \det(A(i|2)) + a_{i3} \cdot (-1)^{i+3} \det(A(i|3)) + \dots \\ + a_{i\ell} \cdot (-1)^{i+\ell} \det(A(i|\ell)) + \dots + a_{in} \cdot (-1)^{i+n} \det(A(i|n)) \end{array}$$

holds.

3. Imagine $P(3)$ has already been established:—

For each (3×3) -square matrix B , if the (k, ℓ) -th entry of B is denoted by $b_{k\ell}$ for each k, ℓ , then the equalities below hold:—

$$\left\{ \begin{array}{l} \det(B) = b_{11} \det(B(1|1)) - b_{12} \det(B(1|2)) + b_{13} \det(B(1|3)) \quad \text{— } (\#_1), \\ \det(B) = -b_{21} \det(B(2|1)) + b_{22} \det(B(2|2)) - b_{23} \det(B(2|3)) \quad \text{— } (\#_2), \\ \det(B) = b_{31} \det(B(3|1)) - b_{32} \det(B(3|2)) + b_{33} \det(B(3|3)) \quad \text{— } (\#_3). \end{array} \right.$$

Using $P(3)$, we are going to deduce $P(4)$:—

For each (4×4) -square matrix C , if the (k, ℓ) -th entry of B is denoted by $c_{k\ell}$ for each k, ℓ , the equalities below hold:—

$$\left\{ \begin{array}{l} \det(C) = c_{11} \det(C(1|1)) - c_{12} \det(C(1|2)) + c_{13} \det(C(1|3)) - c_{14} \det(C(1|4)) \quad \text{— } (\natural_2), \\ \det(C) = -c_{21} \det(C(2|1)) + c_{22} \det(C(2|2)) - c_{23} \det(C(2|3)) + c_{24} \det(C(2|4)) \quad \text{— } (\natural_2), \\ \det(C) = c_{31} \det(C(3|1)) - c_{32} \det(C(3|2)) + c_{33} \det(C(3|3)) - c_{34} \det(C(3|4)) \quad \text{— } (\natural_3), \\ \det(C) = -c_{41} \det(C(4|1)) + c_{42} \det(C(4|2)) - c_{43} \det(C(4|3)) + c_{44} \det(C(4|4)) \quad \text{— } (\natural_4). \end{array} \right.$$

Suppose $C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix}$.

By definition, the equality (\natural_1) holds:—

$$\det(C) = c_{11} \det(C(1|1)) - c_{12} \det(C(1|2)) + c_{13} \det(C(1|3)) - c_{14} \det(C(1|4))$$

We verify these equalities:—

$$\left\{ \begin{array}{l} \det(C) = -c_{21} \det(C(2|1)) + c_{22} \det(C(2|2)) - c_{23} \det(C(2|3)) + c_{24} \det(C(2|4)) \quad \text{— } (\natural_2), \\ \det(C) = c_{31} \det(C(3|1)) - c_{32} \det(C(3|2)) + c_{33} \det(C(3|3)) - c_{34} \det(C(3|4)) \quad \text{— } (\natural_3), \\ \det(C) = -c_{41} \det(C(4|1)) + c_{42} \det(C(4|2)) - c_{43} \det(C(4|3)) + c_{44} \det(C(4|4)) \quad \text{— } (\natural_4). \end{array} \right.$$

We verify (b₂):

$$\begin{aligned}
& \det(C) = c_{11} \det(C(1|1)) - c_{12} \det(C(1|2)) + c_{13} \det(C(1|3)) - c_{14} \det(C(1|4)) \\
&= c_{11} \det \begin{pmatrix} c_{22} & c_{23} & c_{24} \\ c_{32} & c_{33} & c_{34} \\ c_{42} & c_{43} & c_{44} \end{pmatrix} - c_{12} \det \begin{pmatrix} c_{21} & c_{23} & c_{24} \\ c_{31} & c_{33} & c_{34} \\ c_{41} & c_{43} & c_{44} \end{pmatrix} \\
&\quad + c_{13} \det \begin{pmatrix} c_{21} & c_{22} & c_{24} \\ c_{31} & c_{32} & c_{34} \\ c_{41} & c_{42} & c_{44} \end{pmatrix} - c_{14} \det \begin{pmatrix} c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \\ c_{41} & c_{42} & c_{43} \end{pmatrix} \\
&= c_{11} \left(c_{22} \det \begin{pmatrix} c_{33} & c_{34} \\ c_{43} & c_{44} \end{pmatrix} - c_{23} \det \begin{pmatrix} c_{32} & c_{34} \\ c_{42} & c_{44} \end{pmatrix} + c_{24} \det \begin{pmatrix} c_{32} & c_{33} \\ c_{42} & c_{43} \end{pmatrix} \right) \\
&\quad - c_{12} \left(c_{21} \det \begin{pmatrix} c_{33} & c_{34} \\ c_{43} & c_{44} \end{pmatrix} - c_{23} \det \begin{pmatrix} c_{31} & c_{34} \\ c_{41} & c_{44} \end{pmatrix} + c_{24} \det \begin{pmatrix} c_{31} & c_{33} \\ c_{41} & c_{43} \end{pmatrix} \right) \\
&\quad + c_{13} \left(c_{21} \det \begin{pmatrix} c_{32} & c_{34} \\ c_{42} & c_{44} \end{pmatrix} - c_{22} \det \begin{pmatrix} c_{31} & c_{34} \\ c_{41} & c_{44} \end{pmatrix} + c_{24} \det \begin{pmatrix} c_{31} & c_{32} \\ c_{41} & c_{42} \end{pmatrix} \right) \\
&\quad - c_{14} \left(c_{21} \det \begin{pmatrix} c_{32} & c_{33} \\ c_{42} & c_{43} \end{pmatrix} - c_{22} \det \begin{pmatrix} c_{31} & c_{33} \\ c_{41} & c_{43} \end{pmatrix} + c_{23} \det \begin{pmatrix} c_{31} & c_{32} \\ c_{41} & c_{42} \end{pmatrix} \right) \\
&= -c_{21} \left(c_{12} \det \begin{pmatrix} c_{33} & c_{34} \\ c_{43} & c_{44} \end{pmatrix} - c_{13} \det \begin{pmatrix} c_{32} & c_{34} \\ c_{42} & c_{44} \end{pmatrix} + c_{14} \det \begin{pmatrix} c_{32} & c_{33} \\ c_{42} & c_{43} \end{pmatrix} \right) \\
&\quad + c_{22} \left(c_{11} \det \begin{pmatrix} c_{33} & c_{34} \\ c_{43} & c_{44} \end{pmatrix} - c_{13} \det \begin{pmatrix} c_{31} & c_{34} \\ c_{41} & c_{44} \end{pmatrix} + c_{14} \det \begin{pmatrix} c_{31} & c_{33} \\ c_{41} & c_{43} \end{pmatrix} \right) \\
&\quad - c_{23} \left(c_{11} \det \begin{pmatrix} c_{32} & c_{34} \\ c_{42} & c_{44} \end{pmatrix} - c_{12} \det \begin{pmatrix} c_{31} & c_{34} \\ c_{41} & c_{44} \end{pmatrix} + c_{14} \det \begin{pmatrix} c_{31} & c_{32} \\ c_{41} & c_{42} \end{pmatrix} \right) \\
&\quad + c_{24} \left(c_{11} \det \begin{pmatrix} c_{32} & c_{33} \\ c_{42} & c_{43} \end{pmatrix} - c_{12} \det \begin{pmatrix} c_{31} & c_{33} \\ c_{41} & c_{43} \end{pmatrix} + c_{13} \det \begin{pmatrix} c_{31} & c_{32} \\ c_{41} & c_{42} \end{pmatrix} \right) \\
&= -c_{21} \det \begin{pmatrix} c_{12} & c_{13} & c_{14} \\ c_{32} & c_{33} & c_{34} \\ c_{42} & c_{43} & c_{44} \end{pmatrix} + c_{22} \det \begin{pmatrix} c_{11} & c_{13} & c_{14} \\ c_{31} & c_{33} & c_{34} \\ c_{41} & c_{43} & c_{44} \end{pmatrix} \\
&\quad - c_{23} \det \begin{pmatrix} c_{11} & c_{12} & c_{14} \\ c_{31} & c_{32} & c_{34} \\ c_{41} & c_{42} & c_{44} \end{pmatrix} + c_{24} \det \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{31} & c_{32} & c_{33} \\ c_{41} & c_{42} & c_{43} \end{pmatrix} \\
&= -c_{21} \det(C(2|1)) + c_{22} \det(C(2|2)) - c_{23} \det(C(2|3)) + c_{24} \det(C(2|4))
\end{aligned}$$

We verify (4₃):

$$\begin{aligned}
& \det(C) = c_{11} \det(C(1|1)) - c_{12} \det(C(1|2)) + c_{13} \det(C(1|3)) - c_{14} \det(C(1|4)) \\
&= c_{11} \det \begin{pmatrix} c_{22} & c_{23} & c_{24} \\ c_{32} & c_{33} & c_{34} \\ c_{42} & c_{43} & c_{44} \end{pmatrix} - c_{12} \det \begin{pmatrix} c_{21} & c_{23} & c_{24} \\ c_{31} & c_{33} & c_{34} \\ c_{41} & c_{43} & c_{44} \end{pmatrix} \\
&\quad + c_{13} \det \begin{pmatrix} c_{21} & c_{22} & c_{24} \\ c_{31} & c_{32} & c_{34} \\ c_{41} & c_{42} & c_{44} \end{pmatrix} - c_{14} \det \begin{pmatrix} c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \\ c_{41} & c_{42} & c_{43} \end{pmatrix} \\
&= c_{11} \left(-c_{32} \det \begin{pmatrix} c_{23} & c_{24} \\ c_{43} & c_{44} \end{pmatrix} + c_{33} \det \begin{pmatrix} c_{22} & c_{24} \\ c_{42} & c_{44} \end{pmatrix} - c_{34} \det \begin{pmatrix} c_{22} & c_{23} \\ c_{42} & c_{43} \end{pmatrix} \right) \\
&\quad - c_{12} \left(-c_{31} \det \begin{pmatrix} c_{23} & c_{24} \\ c_{43} & c_{44} \end{pmatrix} + c_{33} \det \begin{pmatrix} c_{21} & c_{24} \\ c_{41} & c_{44} \end{pmatrix} - c_{34} \det \begin{pmatrix} c_{21} & c_{23} \\ c_{41} & c_{43} \end{pmatrix} \right) \\
&\quad + c_{13} \left(-c_{31} \det \begin{pmatrix} c_{22} & c_{24} \\ c_{42} & c_{44} \end{pmatrix} + c_{32} \det \begin{pmatrix} c_{21} & c_{24} \\ c_{41} & c_{44} \end{pmatrix} - c_{34} \det \begin{pmatrix} c_{21} & c_{22} \\ c_{41} & c_{42} \end{pmatrix} \right) \\
&\quad - c_{14} \left(-c_{31} \det \begin{pmatrix} c_{22} & c_{23} \\ c_{42} & c_{43} \end{pmatrix} + c_{32} \det \begin{pmatrix} c_{21} & c_{23} \\ c_{41} & c_{43} \end{pmatrix} - c_{33} \det \begin{pmatrix} c_{21} & c_{22} \\ c_{41} & c_{42} \end{pmatrix} \right) \\
&= c_{31} \left(c_{12} \det \begin{pmatrix} c_{23} & c_{24} \\ c_{43} & c_{44} \end{pmatrix} - c_{13} \det \begin{pmatrix} c_{22} & c_{24} \\ c_{42} & c_{44} \end{pmatrix} + c_{14} \det \begin{pmatrix} c_{22} & c_{23} \\ c_{42} & c_{43} \end{pmatrix} \right) \\
&\quad - c_{32} \left(c_{11} \det \begin{pmatrix} c_{23} & c_{24} \\ c_{43} & c_{44} \end{pmatrix} - c_{13} \det \begin{pmatrix} c_{21} & c_{24} \\ c_{41} & c_{44} \end{pmatrix} + c_{14} \det \begin{pmatrix} c_{21} & c_{23} \\ c_{41} & c_{43} \end{pmatrix} \right) \\
&\quad + c_{33} \left(c_{11} \det \begin{pmatrix} c_{22} & c_{24} \\ c_{42} & c_{44} \end{pmatrix} - c_{12} \det \begin{pmatrix} c_{21} & c_{24} \\ c_{41} & c_{44} \end{pmatrix} + c_{14} \det \begin{pmatrix} c_{21} & c_{22} \\ c_{41} & c_{42} \end{pmatrix} \right) \\
&\quad - c_{34} \left(c_{11} \det \begin{pmatrix} c_{22} & c_{23} \\ c_{42} & c_{43} \end{pmatrix} - c_{12} \det \begin{pmatrix} c_{21} & c_{23} \\ c_{41} & c_{43} \end{pmatrix} + c_{13} \det \begin{pmatrix} c_{21} & c_{22} \\ c_{41} & c_{42} \end{pmatrix} \right) \\
&= c_{31} \det \begin{pmatrix} c_{12} & c_{13} & c_{14} \\ c_{22} & c_{23} & c_{24} \\ c_{42} & c_{43} & c_{44} \end{pmatrix} - c_{32} \det \begin{pmatrix} c_{11} & c_{13} & c_{14} \\ c_{21} & c_{23} & c_{24} \\ c_{41} & c_{43} & c_{44} \end{pmatrix} \\
&\quad + c_{33} \det \begin{pmatrix} c_{11} & c_{12} & c_{14} \\ c_{21} & c_{22} & c_{24} \\ c_{41} & c_{42} & c_{44} \end{pmatrix} - c_{34} \det \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{41} & c_{42} & c_{43} \end{pmatrix} \\
&= c_{31} \det(C(3|1)) - c_{32} \det(C(3|2)) + c_{33} \det(C(3|3)) - c_{34} \det(C(3|4))
\end{aligned}$$

We verify (4):

$$\begin{aligned}
& \det(C) = c_{11} \det(C(1|1)) - c_{12} \det(C(1|2)) + c_{13} \det(C(1|3)) - c_{14} \det(C(1|4)) \\
&= c_{11} \det \begin{pmatrix} c_{22} & c_{23} & c_{24} \\ c_{32} & c_{33} & c_{34} \\ c_{42} & c_{43} & c_{44} \end{pmatrix} - c_{12} \det \begin{pmatrix} c_{21} & c_{23} & c_{24} \\ c_{31} & c_{33} & c_{34} \\ c_{41} & c_{43} & c_{44} \end{pmatrix} \\
&\quad + c_{13} \det \begin{pmatrix} c_{21} & c_{22} & c_{24} \\ c_{31} & c_{32} & c_{34} \\ c_{41} & c_{42} & c_{44} \end{pmatrix} - c_{14} \det \begin{pmatrix} c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \\ c_{41} & c_{42} & c_{43} \end{pmatrix} \\
&= c_{11} \left(c_{42} \det \begin{pmatrix} c_{23} & c_{24} \\ c_{33} & c_{34} \end{pmatrix} - c_{43} \det \begin{pmatrix} c_{22} & c_{24} \\ c_{32} & c_{34} \end{pmatrix} + c_{44} \det \begin{pmatrix} c_{22} & c_{23} \\ c_{32} & c_{33} \end{pmatrix} \right) \\
&\quad - c_{12} \left(c_{41} \det \begin{pmatrix} c_{23} & c_{24} \\ c_{33} & c_{34} \end{pmatrix} - c_{43} \det \begin{pmatrix} c_{21} & c_{24} \\ c_{31} & c_{34} \end{pmatrix} + c_{44} \det \begin{pmatrix} c_{21} & c_{23} \\ c_{31} & c_{33} \end{pmatrix} \right) \\
&\quad + c_{13} \left(c_{41} \det \begin{pmatrix} c_{22} & c_{24} \\ c_{32} & c_{34} \end{pmatrix} - c_{42} \det \begin{pmatrix} c_{21} & c_{24} \\ c_{31} & c_{34} \end{pmatrix} + c_{44} \det \begin{pmatrix} c_{21} & c_{22} \\ c_{31} & c_{32} \end{pmatrix} \right) \\
&\quad - c_{14} \left(c_{41} \det \begin{pmatrix} c_{22} & c_{23} \\ c_{32} & c_{33} \end{pmatrix} - c_{42} \det \begin{pmatrix} c_{21} & c_{23} \\ c_{31} & c_{33} \end{pmatrix} + c_{43} \det \begin{pmatrix} c_{21} & c_{22} \\ c_{31} & c_{32} \end{pmatrix} \right) \\
&= -c_{41} \left(c_{12} \det \begin{pmatrix} c_{23} & c_{24} \\ c_{33} & c_{34} \end{pmatrix} - c_{13} \det \begin{pmatrix} c_{22} & c_{24} \\ c_{32} & c_{34} \end{pmatrix} + c_{14} \det \begin{pmatrix} c_{22} & c_{23} \\ c_{32} & c_{33} \end{pmatrix} \right) \\
&\quad + c_{42} \left(c_{11} \det \begin{pmatrix} c_{23} & c_{24} \\ c_{33} & c_{34} \end{pmatrix} - c_{13} \det \begin{pmatrix} c_{21} & c_{24} \\ c_{31} & c_{34} \end{pmatrix} + c_{14} \det \begin{pmatrix} c_{21} & c_{23} \\ c_{31} & c_{33} \end{pmatrix} \right) \\
&\quad - c_{43} \left(c_{11} \det \begin{pmatrix} c_{22} & c_{24} \\ c_{32} & c_{34} \end{pmatrix} - c_{12} \det \begin{pmatrix} c_{21} & c_{24} \\ c_{31} & c_{34} \end{pmatrix} + c_{14} \det \begin{pmatrix} c_{21} & c_{22} \\ c_{31} & c_{32} \end{pmatrix} \right) \\
&\quad + c_{44} \left(c_{11} \det \begin{pmatrix} c_{22} & c_{23} \\ c_{32} & c_{33} \end{pmatrix} - c_{12} \det \begin{pmatrix} c_{21} & c_{23} \\ c_{31} & c_{33} \end{pmatrix} + c_{13} \det \begin{pmatrix} c_{21} & c_{22} \\ c_{31} & c_{32} \end{pmatrix} \right) \\
&= -c_{41} \det \begin{pmatrix} c_{12} & c_{13} & c_{14} \\ c_{22} & c_{23} & c_{24} \\ c_{32} & c_{33} & c_{34} \end{pmatrix} + c_{42} \det \begin{pmatrix} c_{11} & c_{13} & c_{14} \\ c_{21} & c_{23} & c_{24} \\ c_{31} & c_{33} & c_{34} \end{pmatrix} \\
&\quad - c_{43} \det \begin{pmatrix} c_{11} & c_{12} & c_{14} \\ c_{21} & c_{22} & c_{24} \\ c_{31} & c_{32} & c_{34} \end{pmatrix} + c_{44} \det \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \\
&= -c_{41} \det(C(4|1)) + c_{42} \det(C(4|2)) - c_{43} \det(C(4|3)) + c_{44} \det(C(4|4))
\end{aligned}$$