## 5.1 Determinants of square matrices.

0. *Assumed background.*

   - What has been covered in Topics 1-3.

   *Abstract.* We introduce:—

   - the notion of determinants for square matrices, and the 'expansion of determinants along (arbitrary) rows',
   - some results on determinants for several special types of square matrices which can be evaluated directly by 'expanding along rows', and
   - the multi-linearity and alternating properties in rows for determinants.

   The key idea in the mathematical induction argument for the result on the validity of 'expanding' a determinant along arbitrary rows is illustrated in the *appendix*.

1. **Definition. (Sub-matrix of a square matrix resultant from simultaneous row-and-column deletion.)**

   *Let $A$ be an $(n \times n)$-square matrix.*

   *For each $k, \ell = 1, 2, \cdots, n$, the $((n-1) \times (n-1))$-matrix of $A$ resultant from the simultaneous deletion of its $k$-th row and $\ell$-th column is called the $(k, \ell)$-th* **sub-matrix of** $A$ **(resultant from simultaneous row-and-column deletion)**. *Such a matrix is denoted by $A(k|\ell)$.*

   **Remark.** Where there is no possibility of confusion, we may choose to omit the phrase '*resultant from simultaneous row-and-column deletion*'.

2. **Example (1). (Illustrations of sub-matrices resultant from simultaneous deletion of rows and columns, for square matrices of small sizes.)**

   (a) Suppose $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Then

   $$A(1|1) = \begin{bmatrix} & \\ & a_{22} \end{bmatrix}, \quad A(1|2) = \begin{bmatrix} & \\ a_{21} & \end{bmatrix}, \quad A(2|1) = \begin{bmatrix} & a_{12} \\ & \end{bmatrix}, \quad A(2|2) = \begin{bmatrix} a_{11} & \\ & \end{bmatrix}.$$

   (b) Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then

   $$A(1|1) = \begin{bmatrix} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{bmatrix}, \quad A(1|2) = \begin{bmatrix} & & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{bmatrix}, \quad A(1|3) = \begin{bmatrix} & & \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{bmatrix},$$

   $$A(2|1) = \begin{bmatrix} & a_{12} & a_{13} \\ & & \\ & a_{32} & a_{33} \end{bmatrix}, \quad A(2|2) = \begin{bmatrix} a_{11} & & a_{13} \\ & & \\ a_{31} & & a_{33} \end{bmatrix}, \quad A(2|3) = \begin{bmatrix} a_{11} & a_{12} & \\ & & \\ a_{31} & a_{32} & \end{bmatrix},$$

   $$A(3|1) = \begin{bmatrix} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ & & \end{bmatrix}, \quad A(3|2) = \begin{bmatrix} a_{11} & & a_{13} \\ a_{21} & & a_{23} \\ & & \end{bmatrix}, \quad A(3|3) = \begin{bmatrix} a_{11} & a_{12} & \\ a_{21} & a_{22} & \\ & & \end{bmatrix}$$

   (c) Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$. Then

   $$A(1|1) = \begin{bmatrix} & & & \\ & a_{22} & a_{23} & a_{24} \\ & a_{32} & a_{33} & a_{34} \\ & a_{42} & a_{43} & a_{44} \end{bmatrix}, \quad A(1|2) = \begin{bmatrix} & & & \\ a_{21} & & a_{23} & a_{24} \\ a_{31} & & a_{33} & a_{34} \\ a_{41} & & a_{43} & a_{44} \end{bmatrix}, \quad A(1|3) = \begin{bmatrix} & & & \\ a_{21} & a_{22} & & a_{24} \\ a_{31} & a_{32} & & a_{34} \\ a_{41} & a_{42} & & a_{44} \end{bmatrix}, \quad A(1|4) = \begin{bmatrix} & & & \\ a_{21} & a_{22} & a_{23} & \\ a_{31} & a_{32} & a_{33} & \\ a_{41} & a_{42} & a_{43} & \end{bmatrix},$$

   $$A(2|1) = \begin{bmatrix} & a_{12} & a_{13} & a_{14} \\ & & & \\ & a_{32} & a_{33} & a_{34} \\ & a_{42} & a_{43} & a_{44} \end{bmatrix}, \quad A(2|2) = \begin{bmatrix} a_{11} & & a_{13} & a_{14} \\ & & & \\ a_{31} & & a_{33} & a_{34} \\ a_{41} & & a_{43} & a_{44} \end{bmatrix}, \quad A(2|3) = \begin{bmatrix} a_{11} & a_{12} & & a_{14} \\ & & & \\ a_{31} & a_{32} & & a_{34} \\ a_{41} & a_{42} & & a_{44} \end{bmatrix}, \quad A(2|4) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \\ & & & \\ a_{31} & a_{32} & a_{33} & \\ a_{41} & a_{42} & a_{43} & \end{bmatrix},$$

   $$A(3|1) = \begin{bmatrix} & a_{12} & a_{13} & a_{14} \\ & a_{22} & a_{23} & a_{24} \\ & & & \\ & a_{42} & a_{43} & a_{44} \end{bmatrix}, \quad A(3|2) = \begin{bmatrix} a_{11} & & a_{13} & a_{14} \\ a_{21} & & a_{23} & a_{24} \\ & & & \\ a_{41} & & a_{43} & a_{44} \end{bmatrix}, \quad A(3|3) = \begin{bmatrix} a_{11} & a_{12} & & a_{14} \\ a_{21} & a_{22} & & a_{24} \\ & & & \\ a_{41} & a_{42} & & a_{44} \end{bmatrix}, \quad A(3|4) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \\ a_{21} & a_{22} & a_{23} & \\ & & & \\ a_{41} & a_{42} & a_{43} & \end{bmatrix},$$

   $$A(4|1) = \begin{bmatrix} & a_{12} & a_{13} & a_{14} \\ & a_{22} & a_{23} & a_{24} \\ & a_{32} & a_{33} & a_{34} \\ & & & \end{bmatrix}, \quad A(4|2) = \begin{bmatrix} a_{11} & & a_{13} & a_{14} \\ a_{21} & & a_{23} & a_{24} \\ a_{31} & & a_{33} & a_{34} \\ & & & \end{bmatrix}, \quad A(4|3) = \begin{bmatrix} a_{11} & a_{12} & & a_{14} \\ a_{21} & a_{22} & & a_{24} \\ a_{31} & a_{32} & & a_{34} \\ & & & \end{bmatrix}, \quad A(4|4) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \\ a_{21} & a_{22} & a_{23} & \\ a_{31} & a_{32} & a_{33} & \\ & & & \end{bmatrix}$$

3. **Definition. ('Inductive definition' for determinant through 'expansion along the first row'.)**

   (a) *Let $B$ be a $(1 \times 1)$-square matrix, whose only entry is denoted by $b$.*
   *Then we define the* **determinant of** *$B$ to be the number $b$. We write $\det(B) = b$.*

   (b) *(We define the determinants of square matrices of other sizes inductively).*
   *Suppose $n$ is an integer greater than 1.*
   *Suppose $A$ is an $(n \times n)$-square matrix, whose $(i, j)$-th entry is denoted by $a_{ij}$.*
   *Then we define the* **determinant of** *$A$, which we denote by $\det(A)$, inductively by*

$$\begin{aligned}\det(A) = \quad & a_{11} \cdot (-1)^{1+1} \det(A(1|1)) + a_{12} \cdot (-1)^{1+2} \det(A(1|2)) + a_{13} \cdot (-1)^{1+3} \det(A(1|3)) + \cdots\cdots \\ & + a_{1\ell} \cdot (-1)^{1+\ell} \det(A(1|\ell)) + \cdots\cdots + a_{1n} \cdot (-1)^{1+n} \det(A(1|n)).\end{aligned}$$

   *This 'formula' is referred to as the 'expansion of $\det(A)$ along the first row'.*

**Remark on terminologies.** The determinant of a square matrix (whose entries are numbers) is simply a number. However, we very often borrow terminologies from matrices when working with determinants.

By the phrase ***k*-th row/column of the determinant of** $A$, we mean the $k$-th row/column of the matrix $A$ whose determinant we are considering.

4. **Example (2). (Illustration of 'expansion of determinant along the first row' for square matrices of small size.)**

   (a) Suppose $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Then

$$\begin{aligned}\det(A) &= a_{11} \cdot (-1)^{1+1} \det(A(1|1)) + a_{12} \cdot (-1)^{1+2} \det(A(1|2)) \\ &= a_{11} \cdot (-1)^{1+1} \det(\begin{bmatrix} & \\ & a_{22} \end{bmatrix}) + a_{12} \cdot (-1)^{1+2} \det(\begin{bmatrix} & \\ a_{21} & \end{bmatrix}) \\ &= a_{11}a_{22} - a_{12}a_{21}\end{aligned}$$

   (b) Suppose $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$. Then

$$\begin{aligned}& \det(B) \\ = \; & b_{11} \cdot (-1)^{1+1} \det(B(1|1)) + b_{12} \cdot (-1)^{1+2} \det(B(1|2)) + b_{13} \cdot (-1)^{1+3} \det(B(1|3)) \\ = \; & b_{11} \cdot (-1)^{1+1} \det(\begin{bmatrix} & & \\ & b_{22} & b_{23} \\ & b_{32} & b_{33} \end{bmatrix}) + b_{12} \cdot (-1)^{1+2} \det(\begin{bmatrix} & & \\ b_{21} & & b_{23} \\ b_{31} & & b_{33} \end{bmatrix}) + b_{13} \cdot (-1)^{1+3} \det(\begin{bmatrix} & & \\ b_{21} & b_{22} & \\ b_{31} & b_{32} & \end{bmatrix}) \\ = \; & b_{11} \cdot (-1)^{1+1} (b_{22}b_{33} - b_{23}b_{32}) + b_{12} \cdot (-1)^{1+2} (b_{21}b_{33} - b_{23}b_{31}) + b_{13} \cdot (-1)^{1+3} (b_{21}b_{32} - b_{22}b_{31}) \\ = \; & b_{11}b_{22}b_{33} + b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32} - b_{11}b_{23}b_{32} - b_{12}b_{21}b_{33} - b_{13}b_{22}b_{31}\end{aligned}$$

   (c) Suppose $C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix}$. Then

$$\begin{aligned}& \det(C) \\ = \; & c_{11} \cdot (-1)^{1+1} \det(C(1|1)) + c_{12} \cdot (-1)^{1+2} \det(C(1|2)) + c_{13} \cdot (-1)^{1+3} \det(C(1|3)) + c_{14} \cdot (-1)^{1+4} \det(C(1|4)) \\ = \; & c_{11} \cdot (-1)^{1+1} \det(\begin{bmatrix} & & & \\ & c_{22} & c_{23} & c_{24} \\ & c_{32} & c_{33} & c_{34} \\ & c_{42} & c_{43} & c_{44} \end{bmatrix}) + c_{12} \cdot (-1)^{1+2} \det(\begin{bmatrix} & & & \\ c_{21} & & c_{23} & c_{24} \\ c_{31} & & c_{33} & c_{34} \\ c_{41} & & c_{43} & c_{44} \end{bmatrix}) \\ & + c_{13} \cdot (-1)^{1+3} \det(\begin{bmatrix} & & & \\ c_{21} & c_{22} & & c_{24} \\ c_{31} & c_{32} & & c_{34} \\ c_{41} & c_{42} & & c_{44} \end{bmatrix}) + c_{14} \cdot (-1)^{1+4} \det(\begin{bmatrix} & & & \\ c_{21} & c_{22} & c_{23} & \\ c_{31} & c_{32} & c_{33} & \\ c_{41} & c_{42} & c_{43} & \end{bmatrix}) \\ = \; & \cdots\cdots\cdots \\ = \; & c_{11}c_{22}c_{33}c_{44} + c_{11}c_{23}c_{34}c_{42} + c_{11}c_{24}c_{32}c_{43} + c_{12}c_{21}c_{34}c_{43} + c_{12}c_{24}c_{33}c_{41} + c_{12}c_{23}c_{31}c_{44} \\ & + c_{13}c_{24}c_{31}c_{42} + c_{13}c_{21}c_{32}c_{44} + c_{13}c_{22}c_{34}c_{41} + c_{14}c_{23}c_{32}c_{41} + c_{14}c_{22}c_{31}c_{43} + c_{14}c_{21}c_{33}c_{42} \\ & - c_{11}c_{22}c_{34}c_{43} - c_{11}c_{24}c_{33}c_{42} - c_{11}c_{23}c_{32}c_{44} - c_{12}c_{21}c_{33}c_{44} - c_{12}c_{23}c_{34}c_{41} - c_{12}c_{24}c_{31}c_{43} \\ & - c_{13}c_{24}c_{32}c_{41} - c_{13}c_{22}c_{31}c_{44} - c_{13}c_{21}c_{34}c_{42} - c_{14}c_{23}c_{31}c_{42} - c_{14}c_{21}c_{32}c_{43} - c_{14}c_{22}c_{33}c_{41}\end{aligned}$$

5. **Theorem (1). ('Expansion of determinant along arbitrary rows'.)**

*Suppose $A$ is an $(n \times n)$-square matrix, whose $(i,j)$-th entry is denoted by $a_{ij}$.*

*Then, for each $i = 1, 2, \cdots, n$,*

$$\begin{aligned}\det(A) = \quad & a_{i1}\cdot(-1)^{i+1}\det(A(i|1)) + a_{i2}\cdot(-1)^{i+2}\det(A(i|2)) + a_{i3}\cdot(-1)^{i+3}\det(A(i|3)) + \cdots\cdots \\ & + a_{i\ell}\cdot(-1)^{i+\ell}\det(A(i|\ell)) + \cdots\cdots + a_{in}\cdot(-1)^{i+n}\det(A(i|n)).\end{aligned}$$

**Remark on terminology.** The 'formula' in the conclusion is referred to as the 'expansion of $\det(A)$ along the $i$-th row'.

**Proof of Theorem (1).** Omitted. (This argument is a tedious exercise in mathematical induction, but it is not difficult. The key idea in the 'inductive step' is displayed in the *appendix*.)

6. **Example (3). (Explicit display of 'formulae' of 'expansion of determinant along arbitrary rows' for square matrices of small sizes.)**

(a) Suppose $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Then these equalities hold:—

$$\begin{cases} \det(A) = a_{11}\det(A(1|1)) - a_{12}\det(A(1|2)), \\ \det(A) = -a_{21}\det(A(2|1)) + a_{22}\det(A(2|2)) \end{cases}$$

(Each gives rise to $\det(A) = a_{11}a_{22} - a_{12}a_{21}$.)

(b) Suppose $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$. Then these equalities below hold:—

$$\begin{cases} \det(B) = b_{11}\det(B(1|1)) - b_{12}\det(B(1|2)) + b_{13}\det(B(1|3)), \\ \det(B) = -b_{21}\det(B(2|1)) + b_{22}\det(B(2|2)) - b_{23}\det(B(2|3)), \\ \det(B) = b_{31}\det(B(3|1)) - b_{32}\det(B(3|2)) + b_{33}\det(B(3|3)). \end{cases}$$

(c) Suppose $C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix}$. Then these equalities hold:—

$$\begin{cases} \det(C) = c_{11}\det(C(1|1)) - c_{12}\det(C(1|2)) + c_{13}\det(C(1|3)) - c_{14}\det(C(1|4)), \\ \det(C) = -c_{21}\det(C(2|1)) + c_{22}\det(C(2|2)) - c_{23}\det(C(2|3)) + c_{24}\det(C(2|4)), \\ \det(C) = c_{31}\det(C(3|1)) - c_{32}\det(C(3|2)) + c_{33}\det(C(3|3)) - c_{34}\det(C(3|4)), \\ \det(C) = -c_{41}\det(C(4|1)) + c_{42}\det(C(4|2)) - c_{43}\det(C(4|3)) + c_{44}\det(C(4|4)). \end{cases}$$

7. **Example (4). (Illustration of computation of determinants by 'expansion along rows'.)**

(a) Suppose $B = \begin{bmatrix} 1 & 7 & 0 \\ 6 & 9 & 8 \\ 0 & 1 & 5 \end{bmatrix}$. Then:—

$$\begin{aligned}\det(B) = \det(\begin{bmatrix} 1 & 7 & 0 \\ 6 & 9 & 8 \\ 0 & 1 & 5 \end{bmatrix}) &= 1\cdot\det(\begin{bmatrix} & & \\ & 9 & 8 \\ & 1 & 5 \end{bmatrix}) - 7\cdot\det(\begin{bmatrix} & & \\ 6 & & 8 \\ 0 & & 5 \end{bmatrix}) + 0\cdot\det(\begin{bmatrix} & & \\ 6 & 9 & \\ 0 & 1 & \end{bmatrix}) \\ &= 1\cdot(9\cdot 5 - 8\cdot 1) - 7(6\cdot 5 - 8\cdot 0) + 0 = -173.\end{aligned}$$

$$\begin{aligned}\det(B) = \det(\begin{bmatrix} 1 & 7 & 0 \\ 6 & 9 & 8 \\ 0 & 1 & 5 \end{bmatrix}) &= -6\cdot\det(\begin{bmatrix} & 7 & 0 \\ & & \\ & 1 & 5 \end{bmatrix}) + 9\cdot\det(\begin{bmatrix} 1 & & 0 \\ & & \\ 0 & & 5 \end{bmatrix}) - 8\cdot\det(\begin{bmatrix} 1 & 7 & \\ & & \\ 0 & 1 & \end{bmatrix}) \\ &= -6\cdot(7\cdot 5 - 0\cdot 1) + 9(1\cdot 5 - 0\cdot 0) - 8(1\cdot 1 - 7\cdot 0) = -173.\end{aligned}$$

$$\begin{aligned}\det(B) = \det(\begin{bmatrix} 1 & 7 & 0 \\ 6 & 9 & 8 \\ 0 & 1 & 5 \end{bmatrix}) &= 0\cdot\det(\begin{bmatrix} & 7 & 0 \\ & 9 & 8 \\ & & \end{bmatrix}) - 8\cdot\det(\begin{bmatrix} 1 & & 0 \\ 6 & & 8 \\ & & \end{bmatrix}) + 5\cdot\det(\begin{bmatrix} 1 & 7 & \\ 6 & 9 & \\ & & \end{bmatrix}) \\ &= 0 - 8(1\cdot 8 - 0\cdot 6) + 5(1\cdot 9 - 7\cdot 6) = -173.\end{aligned}$$

(b) Suppose $C = \begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix}$. Then:—

$$
\begin{aligned}
\det(C) &= \det(\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix}) \\
&= 1 \cdot \det(\begin{bmatrix} & & & \\ & 5 & 2 & 5 \\ & 9 & 8 & 0 \\ & 9 & 8 & 3 \end{bmatrix}) - 9 \cdot \det(\begin{bmatrix} & & & \\ 0 & & 2 & 5 \\ 1 & & 8 & 0 \\ 1 & & 8 & 3 \end{bmatrix}) + 7 \cdot \det(\begin{bmatrix} & & & \\ 0 & 5 & & 5 \\ 1 & 9 & & 0 \\ 1 & 9 & & 3 \end{bmatrix}) - 7 \cdot \det(\begin{bmatrix} & & & \\ 0 & 5 & 2 & \\ 1 & 9 & 8 & \\ 1 & 9 & 8 & \end{bmatrix}) \\
&= \cdots = 15.
\end{aligned}
$$

$$
\begin{aligned}
\det(C) &= \det(\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix}) \\
&= -0 \cdot \det(\begin{bmatrix} & 9 & 7 & 7 \\ & & & \\ & 9 & 8 & 0 \\ & 9 & 8 & 3 \end{bmatrix}) + 5 \cdot \det(\begin{bmatrix} 1 & & 7 & 7 \\ & & & \\ 1 & & 8 & 0 \\ 1 & & 8 & 3 \end{bmatrix}) - 2 \cdot \det(\begin{bmatrix} 1 & 9 & & 7 \\ & & & \\ 1 & 9 & & 0 \\ 1 & 9 & & 3 \end{bmatrix}) + 5 \cdot \det(\begin{bmatrix} 1 & 9 & 7 & \\ & & & \\ 1 & 9 & 8 & \\ 1 & 9 & 8 & \end{bmatrix}) \\
&= \cdots = 15.
\end{aligned}
$$

$$
\begin{aligned}
\det(C) &= \det(\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix}) \\
&= 1 \cdot \det\det(\begin{bmatrix} & 9 & 7 & 7 \\ & 5 & 2 & 5 \\ & & & \\ & 9 & 8 & 3 \end{bmatrix}) - 9 \cdot \det(\begin{bmatrix} 1 & & 7 & 7 \\ 0 & & 2 & 5 \\ & & & \\ 1 & & 8 & 3 \end{bmatrix}) + 8 \cdot \det(\begin{bmatrix} 1 & 9 & & 7 \\ 0 & 5 & & 5 \\ & & & \\ 1 & 9 & & 3 \end{bmatrix}) - 0 \cdot \det(\begin{bmatrix} 1 & 9 & 7 & \\ 0 & 5 & 2 & \\ & & & \\ 1 & 9 & 8 & \end{bmatrix}) \\
&= \cdots = 15.
\end{aligned}
$$

$$
\begin{aligned}
\det(C) &= \det(\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix}) \\
&= -1 \cdot \det(\begin{bmatrix} & 9 & 7 & 7 \\ & 5 & 2 & 5 \\ & 9 & 8 & 0 \\ & & & \end{bmatrix}) + 9 \cdot \det(\begin{bmatrix} 1 & & 7 & 7 \\ 0 & & 2 & 5 \\ 1 & & 8 & 0 \\ & & & \end{bmatrix}) - 8 \cdot \det(\begin{bmatrix} 1 & 9 & & 7 \\ 0 & 5 & & 5 \\ 1 & 9 & & 0 \\ & & & \end{bmatrix}) + 3 \cdot \det(\begin{bmatrix} 1 & 9 & 7 & \\ 0 & 5 & 2 & \\ 1 & 9 & 8 & \\ & & & \end{bmatrix}) \\
&= \cdots = 15.
\end{aligned}
$$

8. **Theorem (2). (Determinant of a square matrix with an entire row or an entire column of zeros.)**

   *The statements below hold:—*

   (1) *Suppose $A$ is a square matrix with an entire row of zeros. Then $\det(A) = 0$.*

   (2) *Suppose $B$ is a square matrix with an entire column of zeros. Then $\det(B) = 0$.*

9. **Proof of Theorem (2).**

   The statement (1) follows immediately from the 'expansion of determinants along rows'.

   We apply mathematical induction to prove the statement (2).

   Denote by $P(n)$ the proposition below:—

   - *If $C$ is an $(n \times n)$-square matrix with an entire column of zeros, then $\det(C) = 0$.*

   (a) The only $(1 \times 1)$-square matrix with an entire column of zeros is $[0]$, whose determinant is 0.
   Hence $P(1)$ is true.

   (b) Let $k$ be a positive integer. Suppose $P(k)$ is true. (Hence, if $B$ is a $(k \times k)$-square matrix with an entire column of zeros, then $\det(B) = 0$.)
   Pick any $((k+1) \times (k+1))$-square matrix $C$, whose $(i, j)$-th entry is denoted by $c_{ij}$ for each $i, j$.
   Suppose there is an entire column of zeros in $C$, say, its $\ell$-th column.

Expanding $\det(C)$ along its first row, we have

$$\begin{aligned}\det(C) &= c_{11}\cdot\det(C(1|1)) - c_{12}\cdot\det(C(1|2)) + \cdots + c_{1,\ell-1}\cdot(-1)^{\ell}\det(C(1|\ell-1)) \\ &\quad + c_{1\ell}\cdot(-1)^{\ell+1}\det(C(1|\ell)) \\ &\quad + c_{1,\ell+1}\cdot(-1)^{\ell+2}\det(C(1|\ell+1)) + \cdots + c_{1k}\cdot(-1)^{k+1}\det(C(1|k)) + c_{1,k+1}\cdot(-1)^{k+2}\det(C(1|k+1)).\end{aligned}$$

By assumption, we have $c_{1\ell} = 0$.

Also, for each $j = 1, 2, \cdots \ell-1, \ell+1, \cdots, k, k+1$, the $(k \times k)$-matrices $C(1|j)$ has an entire column of zeros ('originated from' the $\ell$-th column of $C$). Then $\det(C(1|j)) = 0$.

Therefore $\det(C) = 0$.

Hence $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for each positive integer $n$.

10. Theorem (2) can be 'upgraded' into the result below, which will turn out to be an important tool for evaluation of determinants of square matrices.

**Theorem (3).**

*Let $C$ be an $(n \times n)$-square matrix whose $(i,j)$-th entry is denoted by $c_{ij}$.*

*Let $k, \ell$ be integers between 1 and $n$.*

*Suppose at least one of the conditions $(\star), (\star')$ is satisfied:—*

($\star$) *Every entry in the $k$-th row of $C$, with perhaps the exception of the $\ell$-th entry, is 0.*

($\star'$) *Every entry in the $\ell$-th column of $C$, with perhaps the exception of the $k$-th entry, is 0.*

*Then $\det(C) = c_{k\ell}\cdot(-1)^{k+\ell}\det(C(k|\ell))$.*

**Remark.** In plain words, Theorem (3) says:—

*If $C$ is a square matrix in which some row/column has at most one non-zero entry which is located at, say, its $(k,\ell)$-th entry, whose value is $\gamma$, then $\det(C) = \gamma\cdot(-1)^{k+\ell}\det(C(k|\ell))$.*

11. **Proof of Theorem (3).**

Let $C$ be an $(n \times n)$-square matrix whose $(i,j)$-th entry is denoted by $c_{ij}$.

Let $k, \ell$ be integers between 1 and $n$.

(a) Suppose $(\star)$ holds: *Every entry in the $k$-th row of $C$, with perhaps the exception of the $\ell$-th entry, is 0.*

Then, expanding $\det(C)$ along its $k$-th row, we obtain the equality $\det(C) = c_{k\ell}\cdot(-1)^{k+\ell}\det(C(k|\ell))$.

(b) Suppose $(\star')$ holds: *Every entry in the $\ell$-th column of $C$, with perhaps the exception of the $k$-th entry, is 0.*

Then, expanding $\det(C)$ along its $k$-th row, we have

$$\begin{aligned}\det(C) &= c_{k1}\cdot(-1)^{k+1}\det(C(k|1)) + c_{k2}\cdot(-1)^{k+2}\det(C(k|2)) + \cdots + c_{k,\ell-1}\cdot(-1)^{k+\ell-1}\det(C(k|\ell-1)) \\ &\quad + c_{k\ell}\cdot(-1)^{k+\ell}\det(C(k|\ell)) \\ &\quad + c_{k,\ell+1}\cdot(-1)^{k+\ell+1}\det(C(k|\ell+1)) + \cdots + c_{k-1,n}\cdot(-1)^{k+n-1}\det(C(k-1|n)) + c_{kn}\cdot(-1)^{k+n}\det(C(k|n)).\end{aligned}$$

By assumption, for each $j = 1, 2, \cdots \ell-1, \ell+1, \cdots, n-1, n$, each of the $((n-1)\times(n-1))$-matrices $C(k|j)$ has an entire column of zeros ('inherited from the $\ell$-th column of $C$).

Hence, by Theorem (2), we have $\det(C) = c_{k\ell}\cdot(-1)^{k+\ell}\det(C(k|\ell))$.

12. Using Theorem (3), we can deduce Theorem (4), which is about determinants of upper/lower triangular matrices.

**Theorem (4). (Determinants of upper/lower triangular matrices.)**

*The statements below hold:—*

(1) *Suppose $A$ is an upper-triangular matrix. Then $\det(A)$ is the product of the diagonal entries of $A$.*

(2) *Suppose $B$ is a lower-triangular matrix. Then $\det(B)$ is the product of the diagonal entries of $B$.*

13. **Proof of Theorem (4).**

Here we prove the statement (1). The argument for the statement (2) is similar to that for the statement (1).

Denote by $P(n)$ the proposition below:—

- *If $C$ is an $(n \times n)$-upper triangular matrix, whose diagonal entries, from left to right, are $c_{11}, c_{22}, \cdots, c_{nn}$ then* $\det(C) = c_{11}c_{22}\cdots c_{nn}$.

(a) $P(1)$ is true by virtue of definition of determinants.

(b) Let $k$ be a positive integer. Suppose $P(k)$ is true. (Hence, if $B$ is a $(k \times k)$-upper triangular matrix, whose diagonal entries, from left to right, are $b_{11}, b_{22}, \cdots, b_{kk}$ then $\det(B) = b_{11}b_{22}\cdots b_{kk}$.)

Pick any $((k+1) \times (k+1))$-upper-triangular matrix $C$, whose $(i, j)$-th entry is denoted by $c_{ij}$ for each $i, j$.

Note that every entry of the bottom row of $C$, with perhaps the exception of the last entry, is 0.

Then, by Theorem (3), we have

$$\det(C) = c_{k+1,k+1} \cdot (-1)^{(k+1)+(k+1)} \det(C(k+1|k+1)) = c_{k+1,k+1} \det(C(k+1|k+1)).$$

Note that the $(k \times k)$-matrices $C(k|k)$ is an upper-triangular matrix, whose diagonal entries, from left to right, are $c_{11}, c_{12}, \cdots, c_{kk}$. Then by $P(k)$, we have $\det(C(k+1|k+1)) = c_{11}c_{22}\cdots c_{kk}$.

Therefore $\det(C) = c_{k+1,k+1} \cdot \det(C(k+1|k+1)) = c_{11}c_{22}\cdots c_{kk}c_{k+1,k+1}$.

Hence $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for each positive integer $n$.

14. **Theorem (5). (Corollary (1) to Theorem (4), about determinants of diagonal matrices.)**

*Suppose $D$ is a diagonal matrix. Then $\det(D)$ is the product of the diagonal entries of $D$.*

*In particular, $\det(I_n) = 1$ for each positive integer $n$.*

15. Determinants of square matrices which are row-echelon forms are easy to compute. But we want to be more thorough than just describing their determinants.

**Theorem (6).**

*Let $J$ be an $(n \times n)$-square matrix. Suppose $J$ is a row-echelon form, of rank $r$. Then:—*

(a) *The first non-zero entry in each non-zero row in $J$ is:*
   - *the diagonal entry in that row, or*
   - *some entry strictly to the right of the diagonal entry in that row.*

(b) *$J$ is an upper-triangular matrix.*

(c) *The statements below are logically equivalent:—*
   i. $J$ is invertible.
   ii. Every column of $J$ is a pivot column.
   iii. Every row of $J$ is a non-zero row.
   iv. The number of rows of $J$, the number of columns in $J$, and the rank of $J$, are equal to each other.
   v. $J$ is an upper-triangular matrix whose diagonal entries are all non-zero.

(d) *$\det(J)$ is the product of the diagonal entries of $J$.*

*Moreover, the statements below hold:—*

(1) *If $r = n$, then $\det(J) \neq 0$.*

(2) *If $r < n$, then $\det(J) = 0$.*

**Remark.** The proof of Theorem (6) is left as an exercise (on invertibility and the application of Theorem (4)).

16. If we only rely on 'expansions of determinants along rows' in evaluating determinant, we will have a hard time. We now introduce a pair of ideas, known as 'multi-linearity' and 'alternating property', that will help us handle determinants not only in computation but also in theoretical discussion.

We start with 'multi-linearity'.

**Theorem (7). (Multi-linearity in rows for determinants.)**

*Let $A, B, C$ be $(n \times n)$-square matrix. For each $i = 1, 2, \cdots, n$, denote the $i$-th rows by $\mathbf{a}_i, \mathbf{b}_i, \mathbf{c}_i$ respectively.*

*Let $p$ be an integer between 1 and $n$. Suppose $\alpha, \beta$ are numbers. and also suppose:—*

(1) $\mathbf{c}_p = \alpha\mathbf{a}_p + \beta\mathbf{b}_p$, *and*

(2) $\mathbf{c}_i = \mathbf{a}_i = \mathbf{b}_i$ *whenever $i \neq p$.*

*Then* $\det(C) = \alpha\det(A) + \beta\det(B)$.

**Remark.** In symbolic terms, the conclusion in Theorem (7) reads:—

$$\det(\left[\begin{array}{c} C_\sharp \\ \hline \alpha\mathbf{a}_p + \beta\mathbf{b}_p \\ \hline C_\flat \end{array}\right]) = \alpha\det(\left[\begin{array}{c} C_\sharp \\ \hline \mathbf{a}_p \\ \hline C_\flat \end{array}\right]) + \beta\det(\left[\begin{array}{c} C_\sharp \\ \hline \mathbf{b}_p \\ \hline C_\flat \end{array}\right]),$$

in which:—

- $C_\sharp$ stands for the matrix whose rows are $\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_{p-1}$ from top to bottom, and
- $C_\flat$ stands for the matrix whose rows are $\mathbf{c}_{p+1}, \cdots, \mathbf{c}_{n-1}, \mathbf{c}_n$ from top to bottom.

Because of the validity of such an equality concerned with the $p$-th row of determinants of square matrices, we say that the determinant is **linear in its $p$-th row**.

Overall, we say the determinant is **multi-linear in its rows**.

17. **Proof of Theorem (7).**

Let $A, B, C$ be $(n \times n)$-square matrix. For each $i = 1, 2, \cdots, n$, denote the $i$-th rows by $\mathbf{a}_i, \mathbf{b}_i, \mathbf{c}_i$ respectively.

Let $p$ be an integer between 1 and $n$. Suppose $\alpha, \beta$ are numbers, and also suppose:—

(1) $\mathbf{c}_p = \alpha\mathbf{a}_p + \beta\mathbf{b}_p$, and

(2) $\mathbf{a}_i = \mathbf{b}_i = \mathbf{c}_i$ whenever $i \neq p$.

Denote the $j$-th entries of $\mathbf{a}_p, \mathbf{b}_p, \mathbf{c}_p$ by $a_{pj}, b_{pj}, c_{pj}$ for each $j$.

By assumption, for each $j = 1, 2, \cdots, n$, we have $c_{pj} = \alpha a_{pj} + \beta b_{pj}$, and $A(p|j) = B(p|j) = C(p|j)$.

Expanding $\det(C)$ along its $p$-th row, we have

$$\begin{aligned}
& \det(C) \\
= \; & c_{p1} \cdot (-1)^{p+1}\det(C(p|1)) + c_{p2} \cdot (-1)^{p+2}\det(C(p|2)) + \cdots + c_{pn} \cdot (-1)^{p+n}\det(C(p|n)) \\
= \; & (\alpha a_{p1} + \beta b_{p1}) \cdot (-1)^{p+1}\det(C(p|1)) + (\alpha a_{p2} + \beta b_{p2}) \cdot (-1)^{p+2}\det(C(p|2)) + \cdots + (\alpha a_{pn} + \beta b_{pn}) \cdot (-1)^{p+n}\det(C(p|n)) \\
= \; & \alpha(a_{p1} \cdot (-1)^{p+1}\det(C(p|1)) + a_{p2} \cdot (-1)^{p+2}\det(C(p|2)) + \cdots + a_{pn} \cdot (-1)^{p+n}\det(C(p|n))) \\
& + \beta(b_{p1} \cdot (-1)^{p+1}\det(C(p|1)) + b_{p2} \cdot (-1)^{p+2}\det(C(p|2)) + \cdots + b_{pn} \cdot (-1)^{p+n}\det(C(p|n))) \\
= \; & \alpha(a_{p1} \cdot (-1)^{p+1}\det(A(p|1)) + a_{p2} \cdot (-1)^{p+2}\det(A(p|2)) + \cdots + a_{pn} \cdot (-1)^{p+n}\det(A(p|n))) \\
& + \beta(b_{p1} \cdot (-1)^{p+1}\det(B(p|1)) + b_{p2} \cdot (-1)^{p+2}\det(B(p|2)) + \cdots + b_{pn} \cdot (-1)^{p+n}\det(B(p|n))) \\
= \; & \alpha\det(A) + \beta\det(B)
\end{aligned}$$

18. We now turn to the 'alternating property'.

**Lemma (8). (Alternating property in neighbouring rows for determinants.)**

*Let $A, B$ be $(n \times n)$-square matrices. For each $i = 1, 2, \cdots, n$, denote the $i$-th rows of $A, B$ by $\mathbf{a}_i, \mathbf{b}_i$ respectively.*

*Suppose $q$ is an integer between 1 and $n - 1$, and suppose:—*

(1) $\mathbf{b}_q = \mathbf{a}_{q+1}$, *and* $\mathbf{b}_{q+1} = \mathbf{a}_q$, *and*

(2) $\mathbf{b}_j = \mathbf{a}_j$ *whenever* $j < q$ *or* $j > q + 1$.

*Then* $\det(B) = -\det(A)$.

**Remark.** In symbolic terms, the conclusion in Lemma (8) reads:—

$$\det(\left[\begin{array}{c} A_\sharp \\ \hline \mathbf{a}_{q+1} \\ \hline \mathbf{a}_q \\ \hline A_\flat \end{array}\right]) = -\det(\left[\begin{array}{c} A_\sharp \\ \hline \mathbf{a}_q \\ \hline \mathbf{a}_{q+1} \\ \hline A_\flat \end{array}\right]),$$

in which:—

- $A_\sharp$ stands for the matrix whose rows are $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_{q-1}$ from top to bottom, and
- $A_\flat$ stands for the matrix whose rows are $\mathbf{a}_{q+2}, \cdots, \mathbf{a}_{n-1}, \mathbf{a}_n$ from top to bottom.

19. **Proof of Lemma (8).**

Let $A, B$ be $(n \times n)$-square matrices. For each $i = 1, 2, \cdots, n$, denote the $i$-th rows of $A, B$ by $\mathbf{a}_i, \mathbf{b}_i$ respectively.

Suppose $q$ is an integer between 1 and $n - 1$, and suppose:—

(1) $\mathbf{b}_q = \mathbf{a}_{q+1}$, *and* $\mathbf{b}_{q+1} = \mathbf{a}_q$, *and*

(2) $\mathbf{b}_j = \mathbf{a}_j$ *whenever* $j < q$ *or* $j > q+1$.

For each $j = 1, 2, \cdots, n$, denote the $j$-th entry of $\mathbf{a}_q$ by $a_{qj}$.

Then for each $j = 1, 2, \cdots, n$, we have $a_{qj} = b_{q+1,j}$, and $A(q|j) = B(q+1|j)$.

Expand $\det(B)$ along the $(q+1)$-th row:

$$\begin{aligned}
& \det(B) \\
= \ & b_{q+1,1} \cdot (-1)^{q+1+1} \det(B(q+1|1)) + b_{q+1,2} \cdot (-1)^{q+1+2} \det(B(q+1|2)) + b_{q+1+3} \cdot (-1)^{q+1+3} \det(B(q+1|3)) \\
& + \cdots + b_{q+1,n} \cdot (-1)^{q+1+n} \det(B(q+1|n)) \\
= \ & a_{q1} \cdot (-1)^{q+2} \det(A(q|1)) + a_{q2} \cdot (-1)^{q+3} \det(A(q|2)) + a_{q3} \cdot (-1)^{q+4} \det(A(q|3)) \\
& + \cdots + a_{qn} \cdot (-1)^{q+1+n} \det(A(q|n)) \\
= \ & -[a_{q1} \cdot (-1)^{q+1} \det(A(q|1)) + a_{q2} \cdot (-1)^{q+2} \det(A(q|2)) + a_{q3} \cdot (-1)^{q+3} \det(A(q|3)) + \cdots + a_{q+n} \cdot (-1)^{q+n} \det(A(q|n))] \\
= \ & -\det(A)
\end{aligned}$$

20. Lemma (8) is needed for proving Theorem (9), and will in fact be superceded by Theorem (9).

**Theorem (9). (Alternating property in rows for determinants.)**

*Let* $A, C$ *be* $(n \times n)$*-square matrices. For each* $i = 1, 2, \cdots, n$*, denote the* $i$*-th rows of* $A, C$ *by* $\mathbf{a}_i, \mathbf{c}_i$ *respectively.*

*Suppose* $p, q$ *are distinct integers amongst* $1, 2, \cdots, n$*, and further suppose*

(a) $\mathbf{c}_q = \mathbf{a}_p$,

(b) $\mathbf{c}_p = \mathbf{a}_q$, *and*

(c) $\mathbf{c}_i = \mathbf{a}_i$ *whenever* $i \neq p$ *and* $i \neq q$.

*Then* $\det(C) = -\det(A)$.

**Remark.** In symbolic terms, the conclusion in Theorem (9) reads:—

$$\det\left(\begin{bmatrix} A_\sharp \\ \hline \mathbf{a}_q \\ \hline A_\natural \\ \hline \mathbf{a}_p \\ \hline A_\flat \end{bmatrix}\right) = -\det\left(\begin{bmatrix} A_\sharp \\ \hline \mathbf{a}_p \\ \hline A_\natural \\ \hline \mathbf{a}_q \\ \hline A_\flat \end{bmatrix}\right),$$

in which:—

- $A_\sharp$ stands for the matrix whose rows are $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_{p-1}$ from top to bottom,
- $A_\natural$ stands for the matrix whose rows are $\mathbf{a}_{p+1}, \mathbf{a}_{p+2}, \cdots, \mathbf{a}_{q-1}$ from top to bottom,
- $A_\flat$ stands for the matrix whose rows are $\mathbf{a}_{q+2}, \cdots, \mathbf{a}_{n-1}, \mathbf{a}_n$ from top to bottom.

Because of the validity of such equalities as described in the conclusion of Theorem (9), we say that the determinant is **alternating in its rows**.

21. An immediately consequence of Theorem (9) is the result below:—

**Theorem (10). (Corollary to Theorem (9), about determinants with identical rows in distinct positions.))**

*Let* $A$ *be a square matrix.*

*Suppose two rows of* $A$ *at distinct positions are identical. Then* $\det(A) = 0$.

**Proof of Theorem (10).**

Let $A$ be a square matrix.

Suppose two distinct rows of $A$, say, the $p$-th row and the $q$-th row, are identical.

Denote by $C$ the square matrix whose $p$-th and $q$-th rows are respectively the $q$-th and $p$-th row of $A$, and whose every other row is the same as the corresponding row of $A$.

By Theorem (9), we have $\det(C) = -\det(A)$.

Also note that $C = A$ by the definition of $C$. Hence $\det(A) = 0$.

22. **Proof of Theorem (9).**

Let $A, C$ be $(n \times n)$-square matrices. For each $i = 1, 2, \cdots, n$, denote the $i$-th rows of $A, C$ by $\mathbf{a}_i, \mathbf{c}_i$ respectively.

Suppose $p, q$ are distinct integers amongst $1, 2, \cdots, n$, and further suppose

(a) $\mathbf{c}_q = \mathbf{a}_p$,

(b) $\mathbf{c}_p = \mathbf{a}_q$, and

(c) $\mathbf{c}_i = \mathbf{a}_i$ whenever $i \neq p$ and $i \neq q$.

Without loss of generality, suppose $p < q$. Write $m = q - p$. So by definition, $q = p + m$.

Note that $A, C$ are row-equivalent under the sequence of row operations below:—

$$\begin{array}{rcl} C & = & B_0 \xrightarrow{R_p \leftrightarrow R_{p+1}} B_1 \xrightarrow{R_{p+1} \leftrightarrow R_{p+2}} B_2 \xrightarrow{R_{p+2} \leftrightarrow R_{p+3}} B_3 \longrightarrow \cdots \longrightarrow B_{m-2} \xrightarrow{R_{q-2} \leftrightarrow R_{q-1}} B_{m-1} \xrightarrow{R_{q-1} \leftrightarrow R_q} B_m \\ & & \xrightarrow{R_{q-1} \leftrightarrow R_{q-2}} B_{m+1} \xrightarrow{R_{q-2} \leftrightarrow R_{q-3}} B_{m+2} \longrightarrow \cdots \longrightarrow B_{2m-3} \xrightarrow{R_{p+2} \leftrightarrow R_{p+1}} B_{2m-2} \xrightarrow{R_{p+1} \leftrightarrow R_p} B_{2m-1} = A \end{array}$$

(in which the matrices $B_1, B_2, \cdots, B_{2m-1}$ are determined by the row operations specified the sequence).

For each $j = 0, 1, 2, \cdots, 2m-2$, the matrices $B_j, B_{j+1}$ distinct from each other only at a pair of neighbouring rows.

Then by Lemma (8), we have $\det(B_{j+1}) = -\det(B_j)$.

Therefore

$$\begin{array}{rcl} \det(C) & = & \det(B_0) = -\det(B_1) = (-1)^2 \det(B_2) \\ & = & \cdots \\ & = & (-1)^{2m-3} \det(B_{2m-3}) = (-1)^{2m-2} \det(B_{2m-2}) = (-1)^{2m-1} \det(B_{2m-1}) = -\det(A). \end{array}$$

23. **Illustration of the idea in the argument for Theorem (9).**

Let $A$ be a $(5 \times 5)$-square matrix, whose rows, from top to bottom, are labelled $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5$.

We verify that

$$\det\left(\begin{bmatrix} \mathbf{a}_5 \\ \hline \mathbf{a}_2 \\ \hline \mathbf{a}_3 \\ \hline \mathbf{a}_4 \\ \hline \mathbf{a}_1 \end{bmatrix}\right) = -\det\left(\begin{bmatrix} \mathbf{a}_1 \\ \hline \mathbf{a}_2 \\ \hline \mathbf{a}_3 \\ \hline \mathbf{a}_4 \\ \hline \mathbf{a}_5 \end{bmatrix}\right).$$

We have

$$\begin{bmatrix} \mathbf{a}_5 \\ \hline \mathbf{a}_2 \\ \hline \mathbf{a}_3 \\ \hline \mathbf{a}_4 \\ \hline \mathbf{a}_1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} \mathbf{a}_2 \\ \hline \mathbf{a}_5 \\ \hline \mathbf{a}_3 \\ \hline \mathbf{a}_4 \\ \hline \mathbf{a}_1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} \mathbf{a}_2 \\ \hline \mathbf{a}_3 \\ \hline \mathbf{a}_5 \\ \hline \mathbf{a}_4 \\ \hline \mathbf{a}_1 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} \mathbf{a}_2 \\ \hline \mathbf{a}_3 \\ \hline \mathbf{a}_4 \\ \hline \mathbf{a}_5 \\ \hline \mathbf{a}_1 \end{bmatrix} \xrightarrow{R_4 \leftrightarrow R_5} \begin{bmatrix} \mathbf{a}_2 \\ \hline \mathbf{a}_3 \\ \hline \mathbf{a}_4 \\ \hline \mathbf{a}_1 \\ \hline \mathbf{a}_5 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} \mathbf{a}_2 \\ \hline \mathbf{a}_3 \\ \hline \mathbf{a}_1 \\ \hline \mathbf{a}_4 \\ \hline \mathbf{a}_5 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} \mathbf{a}_2 \\ \hline \mathbf{a}_1 \\ \hline \mathbf{a}_3 \\ \hline \mathbf{a}_4 \\ \hline \mathbf{a}_5 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} \mathbf{a}_1 \\ \hline \mathbf{a}_2 \\ \hline \mathbf{a}_3 \\ \hline \mathbf{a}_4 \\ \hline \mathbf{a}_5 \end{bmatrix}$$

Hence by Lemma (8):—

$$\begin{array}{rcl} \det\left(\begin{bmatrix} \mathbf{a}_5 \\ \hline \mathbf{a}_2 \\ \hline \mathbf{a}_3 \\ \hline \mathbf{a}_4 \\ \hline \mathbf{a}_1 \end{bmatrix}\right) & = & (-1) \cdot \det\left(\begin{bmatrix} \mathbf{a}_2 \\ \hline \mathbf{a}_5 \\ \hline \mathbf{a}_3 \\ \hline \mathbf{a}_4 \\ \hline \mathbf{a}_1 \end{bmatrix}\right) = (-1)^2 \det\left(\begin{bmatrix} \mathbf{a}_2 \\ \hline \mathbf{a}_3 \\ \hline \mathbf{a}_5 \\ \hline \mathbf{a}_4 \\ \hline \mathbf{a}_1 \end{bmatrix}\right) = (-1)^3 \det\left(\begin{bmatrix} \mathbf{a}_2 \\ \hline \mathbf{a}_3 \\ \hline \mathbf{a}_4 \\ \hline \mathbf{a}_5 \\ \hline \mathbf{a}_1 \end{bmatrix}\right) = (-1)^4 \det\left(\begin{bmatrix} \mathbf{a}_2 \\ \hline \mathbf{a}_3 \\ \hline \mathbf{a}_4 \\ \hline \mathbf{a}_1 \\ \hline \mathbf{a}_5 \end{bmatrix}\right) \\ & = & (-1)^5 \det\left(\begin{bmatrix} \mathbf{a}_2 \\ \hline \mathbf{a}_3 \\ \hline \mathbf{a}_1 \\ \hline \mathbf{a}_4 \\ \hline \mathbf{a}_5 \end{bmatrix}\right) = (-1)^6 \det\left(\begin{bmatrix} \mathbf{a}_2 \\ \hline \mathbf{a}_1 \\ \hline \mathbf{a}_3 \\ \hline \mathbf{a}_4 \\ \hline \mathbf{a}_5 \end{bmatrix}\right) = (-1)^7 \det\left(\begin{bmatrix} \mathbf{a}_1 \\ \hline \mathbf{a}_2 \\ \hline \mathbf{a}_3 \\ \hline \mathbf{a}_4 \\ \hline \mathbf{a}_5 \end{bmatrix}\right) = -\det\left(\begin{bmatrix} \mathbf{a}_1 \\ \hline \mathbf{a}_2 \\ \hline \mathbf{a}_3 \\ \hline \mathbf{a}_4 \\ \hline \mathbf{a}_5 \end{bmatrix}\right) \end{array}$$