4.8.1 Answers to Exercise.

1. (a) Every basis for $\mathcal{W}$ contains three column vectors belonging to $\mathbb{R}^5$. But there are fewer than three column vectors in the list $\mathbf{v}_1, \mathbf{v}_2$.

   (b) Every basis for $\mathcal{W}$ contains three column vectors belonging to $\mathbb{R}^5$. But there are more than three column vectors in the list $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$.

   (c) Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ constitute a basis for $\mathcal{W}$ over the reals. Since $\mathbf{v}_4 \in \mathcal{W}$, $\mathbf{v}_4$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ over the reals, with respect to real scalars say, $\alpha_1, \alpha_2, \alpha_3$.

   Write $\mathbf{t} = \begin{bmatrix} -\alpha_1 \\ -\alpha_2 \\ -\alpha_3 \\ 1 \end{bmatrix}$. Note that $\mathbf{t}$ is a non-zero column vector belonging to $\mathbb{R}^4$.

   We have $V\mathbf{t} = -\alpha_1\mathbf{v}_1 - \alpha_2\mathbf{v}_2 - \alpha_3\mathbf{v}_3 + 1 \cdot \mathbf{v}_4 = \mathbf{0}_5$.
   Hence $\mathbf{t} \in \mathcal{N}(V)$.

2. (a) ——

   (b) ——

   (c) *Comment.*

   It suffices to verify the statement ($\natural$): *For any* $\mathbf{u} \in \mathbb{R}^6$, *for any* $\alpha \in \mathbb{R}$, *if* $\mathbf{u} \in \mathcal{W}$, *then* $\alpha\mathbf{u} \in \mathcal{W}$.

3. ——

4. (a) i. True.
   ii. True.
   iii. True.
   iv. True.
   v. False.

   (b) i. ——

   ii. Take $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\alpha = -1$. Then $\mathbf{u} \in S$ and $\alpha\mathbf{u} \notin S$.

   iii. No. Reason: the statement '*For any* $\mathbf{u} \in \mathbb{R}^3$, *for any* $\alpha \in \mathbb{R}$, *if* $\mathbf{u} \in S$ *then* $\alpha\mathbf{u} \in S$' fails to hold.

5. (a) i. True.
   ii. False.
   iii. False.
   iv. True.
   v. True.

   (b) i. ——

   ii. Take $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Then $\mathbf{u} \in S$ and $\mathbf{v} \in S$ and $\mathbf{u} + \mathbf{v} \notin S$.

   iii. No. The statement '*For any* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, *if* $\mathbf{u} \in S$ *and* $\mathbf{v} \in S$ *then* $\mathbf{u} + \mathbf{v} \in S$' fails to hold.

6. (a) ——

   (b) i. ——
   ii. Yes.
   iii. No.

7. (a) ——

   (b) No.

8. ——

9. (a) No.

(b) Yes.

For any $\mathbf{v} \in \mathbb{R}^3$, if $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ then

$$\mathbf{v} = \frac{3v_1 + v_3}{2}\mathbf{u}_1 + \frac{-13v_1 + 2v_2 - 5v_3}{4}\mathbf{u}_2 + \frac{17v_1 - 2v_2 + 5v_3}{4}\mathbf{u}_3$$

(c) No.

(d) Yes.

For any $\mathbf{v} \in \mathbb{R}^4$, if $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$ then

$$\mathbf{v} = \frac{-4v_1 + v_2 + 2v_3 + v_4}{5}\mathbf{u}_1 + \frac{-3v_1 - 8v_2 + 4v_3 + 2v_4}{5}\mathbf{u}_2 + \frac{3v_1 + 3v_2 + v_3 - 2v_4}{5}\mathbf{u}_3 + \frac{v_1 + v_2 - 3v_3 + v_4}{5}\mathbf{u}_4$$

(e) No.

(f) Yes.

For any $\mathbf{v} \in \mathbb{R}^5$, if $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix}$ then

$$\begin{aligned} \mathbf{v} &= \frac{v_1 + v_2 - v_3 + v_4 - v_5}{2}\mathbf{u}_1 + \frac{-v_1 + v_2 + v_3 - v_4 + v_5}{2}\mathbf{u}_2 + \frac{v_1 - v_2 + v_3 + v_4 - v_5}{2}\mathbf{u}_3 \\ &\quad + \frac{-v_1 + v_2 - v_3 + v_4 + v_5}{2}\mathbf{u}_4 + \frac{v_1 - v_2 + v_3 - v_4 + v_5}{2}\mathbf{u}_5 \end{aligned}$$

(g) No.

10. (In each part, there are many correct choices for the column vectors in the basis concerned.)

(a) $\mathbf{u}_1, \mathbf{u}_2$ constitute a basis for $\mathcal{W}$.
$\dim(\mathcal{W}) = 2$.
$\mathbf{u}_3 = \mathbf{u}_1 + \mathbf{u}_2$, $\mathbf{u}_4 = -4\mathbf{u}_1 + 3\mathbf{u}_2$, $\mathbf{u}_5 = 4\mathbf{u}_1 - \mathbf{u}_2$.
Reason:

$$\left[\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4 \mid \mathbf{u}_5\ \right] \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & 0 & 1 & -4 & 4 \\ 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_4$ constitute a basis for $\mathcal{W}$.
$\dim(\mathcal{W}) = 3$.
$\mathbf{u}_3 = 2\mathbf{u}_1 + 3\mathbf{u}_2$, $\mathbf{u}_5 = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_4$.
Reason:

$$\left[\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4 \mid \mathbf{u}_5\ \right] \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(c) $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_4$ constitute a basis for $\mathcal{W}$.
$\dim(\mathcal{W}) = 3$.
$\mathbf{u}_3 = \mathbf{u}_1 - \mathbf{u}_2$, $\mathbf{u}_5 = \mathbf{u}_1 + 3\mathbf{u}_2 + 2\mathbf{u}_4$.
Reason:

$$\left[\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4 \mid \mathbf{u}_5\ \right] \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(d) $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ constitute a basis for $\mathcal{W}$.

$\dim(\mathcal{W}) = 3$.

$\mathbf{u}_4 = 3\mathbf{u}_1 - \mathbf{u}_2 + \mathbf{u}_3$, $\mathbf{u}_5 = 2\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_4$.

Reason:

$$\left[\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4 \mid \mathbf{u}_5\ \right] \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(e) $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_4$ constitute a basis for $\mathcal{W}$.

$\dim(\mathcal{W}) = 3$.

$\mathbf{u}_3 = 2\mathbf{u}_1 + 3\mathbf{u}_2$, $\mathbf{u}_5 = \mathbf{u}_1 - \mathbf{u}_2 + \mathbf{u}_4$, $\mathbf{u}_6 = \mathbf{u}_2 + 2\mathbf{u}_4$.

Reason:

$$\left[\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4 \mid \mathbf{u}_5 \mid \mathbf{u}_6\ \right] \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

11. (a) $3\mathbf{u}$ is a solution of $\mathcal{LS}(A, 3\mathbf{b})$.

(b) $-3\mathbf{u} + \mathbf{v}$ belongs to $\mathcal{N}(A)$.

Alternative answer. Any non-zero scalar multiple of $-3\mathbf{u} + \mathbf{v}$ is equally correct as an answer.

(c) (I): -3. (II): 1.

Alternative answer. (I): $-3c$. (II): c.

Here c stands for any non-zero real number.

(d) $\kappa = -3$ and $\lambda = -4$.

12. (a) 1-st, 2-nd, 4-th, 7-th columns.

(b) 4.

(c) $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4, \mathbf{a}_7$.

(d) i. $\dim(\mathcal{C}(A)) = 4$.

ii. $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4, \mathbf{a}_7$.

iii. $C = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 5 \\ 0 & 0 & 0 & 1 & -1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

$\mathbf{a}_8 = -\mathbf{a}_1 + 5\mathbf{a}_2 + 2\mathbf{a}_4 + 3\mathbf{a}_7$

(e) i. $\dim(\mathcal{N}(A)) = 4$.

ii. $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ 0 \\ -2 \\ 0 \\ 0 \\ -3 \\ 1 \end{bmatrix}$.

(f) i. $\dim(\mathcal{R}(A)) = 4$.

ii. $\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 1 \\ 2 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}$.

iii. $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 2 \\ 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 2 \\ -1 \\ 8 \\ -1 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}.$

iv. $\dim(\mathcal{N}(A^t)) = 2$.

13. (a) B is a reduced row-echelon form.

The pivot columns of B are the first, second, fourth, seventh, eighth columns.

(b) i. $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_4, \mathbf{u}_7, \mathbf{u}_8$ constitute a basis for $\mathcal{C}(A)$.

Remark. There is no alternative answer.

ii. $\dim(\mathcal{C}(A)) = 5$.

(c) i. $\mathcal{R}(B), \mathcal{C}(B^t)$ are equal to $\mathcal{R}(A)$.

ii. A basis for $\mathcal{R}(A)$ is given by

$$\begin{bmatrix} 1 \\ 0 \\ c_{13} \\ 0 \\ c_{15} \\ c_{16} \\ 0 \\ 0 \\ c_{19} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ c_{23} \\ 0 \\ c_{25} \\ c_{26} \\ 0 \\ 0 \\ c_{29} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ c_{35} \\ c_{36} \\ 0 \\ 0 \\ c_{39} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ c_{49} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ c_{59} \end{bmatrix}$$

iii. $\dim(\mathcal{R}(A)) = 5$.

(d) i. $\dim(\mathcal{N}(A^t)) + \dim(\mathcal{C}(A^t)) = 6$.

ii. $\dim(\mathcal{N}(A^t)) = 1$.

14. (a) 1-st, 4-th columns in C are pivot columns.

3-rd, 5-th columns in C are free columns.

$a_{12} = 1, a_{22} = 1, a_{32} = 2, a_{42} = -1, a_{14} = 0, a_{34} = 1, a_{44=}4$.

$c_{11} = 0, c_{21} = 0, c_{31} = 0, c_{12} = 0, c_{22} = 1, c_{32} = 0, c_{14} = 0, c_{24} = 0, c_{33} = 0, c_{34} = 1, c_{35} = 1, c_{16} = 0, c_{26} = 2, c_{36} - 1, c_{46} = 0$

$$A = \begin{bmatrix} 1 & 1 & 3 & 0 & 2 & 2 \\ 0 & 1 & 2 & 1 & 2 & 1 \\ -1 & 2 & 3 & 1 & 2 & 3 \\ 2 & -1 & 0 & 4 & 5 & -6 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) $\dim(\mathcal{C}(A)) = 3$.

$\dim(\mathcal{N}(A)) = 3$.

(c) $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4$ constitute a basis for $\mathcal{C}(A)$.

$\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_6$ consitute a basis for $\mathcal{C}(A)$.

(d) $\dim(\mathcal{R}(A)) = 3$

${\mathbf{d}_1}^t, {\mathbf{d}_2}^t, {\mathbf{d}_3}^t$ constitute a basis for $\mathcal{R}(A)$.

(e) Note that $\mathcal{N}(A^t)$ is of dimension 1, according to the Rank-nullity formulae.

It follows that a basis for $\mathcal{N}(A^t)$ is constituted by exactly one non-zero column vector, say, $\mathbf{v}$, belonging to $\mathbb{R}^4$. Hence any non-trivial solution of the homogeneous system $\mathcal{LS}(A^t, \mathbf{0}_6)$ is a scalar multiple of $\mathbf{v}$. Then any two non-trivial solutions of the homogeneous system $\mathcal{LS}(A^t, \mathbf{0}_6)$ are scalar multiples of each other.

15. (a) $\dim(\mathcal{N}(A)) = 3$.

(b) Yes. Reason:

- $\dim(\mathcal{N}(A)) = 3$.
- $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3 \in \mathcal{N}(A)$.
- $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ are linearly independent.

16. (a) ——

(b) i. $\alpha \neq 1$ if and only if $\mathcal{N}(A_\alpha) \neq \{\mathbf{0}_4\}$.

ii. $\beta = 1$ if and only if $((S_{\beta;c_1,c_2,c_3,c_4})$ is inconsistent for some values of $c_1, c_2, c_3, c_4)$.

17. (a) ——

(b) F can be taken as I_2.

(c) G can be taken to be $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

18. ——

19. (a) ——

(b) $\dim(\mathcal{N}\big(3A(B^2 - 3B)C\big) + \dim(\mathcal{C}\big(2A(B^2 - 3B)C\big)) = 9$.

Comment.

Apply the Rank-nullity Formulae. But first you have to verify that $\mathcal{C}\big(2A(B^2 - 3B)C\big) = \mathcal{C}\big(A(B^2 - 3B)C\big)$ as sets.

20. **Remark**. According to part (b), we may conclude that $\mathcal{V} = \mathcal{W}$ as sets. Overall, the result described by the entire question is an illustration on what is usually known as 'Replacement Theorem' in linear algebra.

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21. (a) $k = 0$.

(b) ——

(c) i. ——

ii. ——

iii. ——

(d) Yes.

Comment.

We have already shown that $\mathbf{x}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{x}$.

It will turn out that $\mathbf{y}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$,

It will also turn out that $\mathbf{v}_4$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{y}$, and hence $\mathbf{v}_4$ is also a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{x}, \mathbf{y}$,

22. (a) Yes.

(b) Yes.

(c) Yes.

(d) Yes.

23. ——

24. True.

25. (a) ——

(b) True.

26. True.

Comment.

Start by picking some basis for $\mathcal{U}$ over the reals, say, $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, and picking some basis for $\mathcal{V}$ over the reals, say, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

27. ——

28. (a) ——

(b) A counter-example is provided by the choice '$A = I_2$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$'.

(c) i. True.

ii. True.

iii. False. A counter-example is provided by the choice '$A = B = \mathcal{O}_{2\times 2}$'.

iv. True.

29. (a) ——

(b) ——

(c) ——

(d) A possible choice is given by $D = \left[\begin{array}{c|c} I_2 & \mathbf{0}_4{}^t \\ \hline \mathbf{0}_4 & J \end{array}\right]$, in which $J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

30. (a) ——

(b) ——