4.8.1 Exercise: Spaces, bases, and dimensions.

1. Let $\mathcal{W}$ be a subspace of $\mathbb{R}^5$ over the reals. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathcal{W}$, and $V = [\ \mathbf{v}_1 \mid \mathbf{v}_2 \mid \mathbf{v}_3 \mid \mathbf{v}_4\ ]$.

   It is known that $\dim(W) = 3$, and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are pairwise distinct.

   (a) Is it possible for $\mathbf{v}_1, \mathbf{v}_2$ to constitute a basis for $\mathcal{W}$ over the reals?
   Justify your answer with reference to the definitions for the notions of basis and dimension.

   (b) Is it possible for $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ to constitute a basis for $\mathcal{W}$?
   Justify your answer with reference to the definitions for the notions of basis and dimension.

   (c) Is the statement ($\sharp$) below true or false?

   ($\sharp$) *Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ constitute a basis for $\mathcal{W}$ over the reals. Then the $\mathcal{N}(V)$ contains a non-zero column vector belonging to $\mathbb{R}^4$.*

   Justify your answer.

2. Let A be a (5×6)-matrix with real entries, and B be a (5×9)-matrix with real entries.

   Suppose $\mathcal{V}$ is a subspace of $\mathbb{R}^9$ over the reals, and

$$\mathcal{W} = \left\{ \ \mathbf{x} \in \mathbb{R}^6 \ \middle| \ \begin{array}{l} \text{There exists some } \mathbf{t} \in \mathcal{V} \\ \text{such that } A\mathbf{x} = B\mathbf{t}. \end{array} \right\}.$$

   (a) Verify that $\mathbf{0}_6 \in \mathcal{W}$.

   (b) Verify the statement ($\sharp$):

   ($\sharp$) *For any $\mathbf{t}, \mathbf{u} \in \mathbb{R}^6$, if $\mathbf{t}, \mathbf{u} \in \mathcal{W}$, then $\mathbf{t} + \mathbf{u} \in \mathcal{W}$.*

   (c) Verify that $\mathcal{W}$ is a subspace of $\mathbb{R}^6$ over the reals.

3. Let A be a $(p \times q)$-matrix with real entries.

   Verify that $\mathcal{C}(A)$ is a subspace of $\mathbb{R}^p$ over the reals, with With direct reference to the definition for the notion of subspace of $\mathbb{R}^n$,

4. Let $S_1 = \left\{ \ \mathbf{x} \in \mathbb{R}^3 \ \middle| \ \text{The sum of the first two entries of } \mathbf{x} \text{ is greater than or equal to the last entry of } \mathbf{x}. \right\}$.

   (a) For each statement below, determine whether it is true of false. Justify your answer.

   i. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in S.$ ii. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in S.$ iii. $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \in S.$ iv. $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \in S.$ v. $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \in S.$

   (b) i. Let $\mathbf{u} \in S$. Denote the j-th entry of $\mathbf{u}$ by u_j for each $j = 1, 2, 3$.
   Verify the statement ($\sharp$).

   ($\sharp$) *Let $\alpha \in \mathbb{R}$. Suppose $\alpha\mathbf{u} \in S$. Then $\alpha \geq 0$.*

   ii. Name some $\mathbf{w} \in S$ and some $\alpha \in \mathbb{R}$ for which $\mathbf{u} \in S$ and $\alpha\mathbf{u} \notin S$, if such exist. Justify your answer.

   iii. Is S a subspace of $\mathbb{R}^3$? Justify your answer.

5. Let $S = \left\{ \ \mathbf{x} \in \mathbb{R}^3 \ \middle| \ \text{The sum of the cubes of the respective entries of } \mathbf{x} \text{ is } 0 \right\}$.

   (a) For each statement below, determine whether it is true of false. Justify your answer.

   i. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in S.$ ii. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in S.$ iii. $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \in S.$ iv. $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \in S.$ v. $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \in S.$

   (b) i. Let $\mathbf{u}, \mathbf{v} \in S$. Denote the respective j-th entries of $\mathbf{u}, \mathbf{v}$ by u_j, v_j for each $j = 1, 2, 3$.
   Verify the statement ($\sharp$):—

($\sharp$) *Suppose* $\mathbf{u}+\mathbf{v} \in S$. *Then* $u_1v_1(u_1+v_1)+u_2v_2(u_2+v_2)+u_3v_3(u_3+v_3)=0$.

ii. Name some $\mathbf{u}, \mathbf{v} \in S$ for which $\mathbf{u}+\mathbf{v} \notin S$, if such exist. Justify your answer.

iii. Is S a subspace of $\mathbb{R}^3$? Justify your answer.

6. Let C be the (4×4)-square matrix given by $C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

Let S be the set given by

$$S = \left\{ \mathbf{x} \in \mathbb{R}^4 \;\middle|\; \mathbf{x}^t C \mathbf{x} = 0. \right\}.$$

(a) Verify the statements $(\sharp_1), (\sharp_2)$:—

($\sharp_1$) $\mathbf{0}_4 \in S$.

($\sharp_2$) *For any* $\mathbf{v} \in \mathbb{R}^4$, *for any* $\alpha \in \mathbb{R}$, *if* $\mathbf{v} \in S$ *then* $\alpha\mathbf{v} \in S$.

(b) i. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^4$. Suppose $\mathbf{w} = \mathbf{u}+\mathbf{v}$.
Verify that $\mathbf{w}^t C \mathbf{w} = \mathbf{u}^t C \mathbf{u} + \mathbf{v}^t C \mathbf{v} + 2\mathbf{u}^t C \mathbf{v}$.

ii. Is it true that $\mathbf{e}_1^{(4)} \in S$? Justify your answer.

iii. Is S a subspace of $\mathbb{R}^4$? Justify your answer.

7. Let C be the (4×4)-square matrix given by $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$.

Let S be the set given by

$$S = \left\{ \mathbf{x} \in \mathbb{R}^4 \;\middle|\; \mathbf{x}^t C \mathbf{x} = 0. \right\}.$$

(a) Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^4$. Suppose $\mathbf{w} = \mathbf{u}+\mathbf{v}$.
Verify that $\mathbf{w}^t C \mathbf{w} = \mathbf{u}^t C \mathbf{u} + \mathbf{v}^t C \mathbf{v} + 2\mathbf{u}^t C \mathbf{v}$.

(b) Is S a subspace of $\mathbb{R}^4$? Justify your answer.

8. For each part below, consider the column vectors belonging to $\mathbb{R}^4$, denoted by $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ here.

With direct reference to the definitions for the notion of set equality and span, verify that $\text{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}) = \mathbb{R}^4$.

(a) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

(b) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

(c) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

(d) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

9. For each part below, consider the column vectors belonging to $\mathbb{R}^n$ (for various vaalues of n) denoted by $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_n$ here.

- Determine whether $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_n$ constitute a basis for $\mathbb{R}^n$.
- Where $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_n$ indeed constitute a basis for $\mathbb{R}^n$, express any arbitrary column vector $\mathbf{v}$ belonging to $\mathbb{R}^n$, whose j-th entry is denoted by v_j for each $j = 1, 2, \cdots, n$, as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_n$ over the reals.

Justify your answer. (You may use whatever characterization of basis for $\mathbb{R}^n$. However, a characterization in terms of invertibility of square matrices may be more convenient, in view of what you are asked to do beyond determining whether $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_n$ constitute a basis for $\mathbb{R}^n$ over the reals.)

(a) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$.

(b) $\mathbf{u}_1 = \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$.

(c) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 3 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 4 \\ 2 \\ 2 \\ 4 \end{bmatrix}$.

(d) $\mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 5 \end{bmatrix}$.

(e) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 3 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 5 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 2 \\ -2 \\ 4 \\ 5 \end{bmatrix}$.

(f) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

(g) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{u}_5 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

10. Consider each of the collection of column vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots$ below. Write $\mathcal{W} = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots\})$.

- Determine the dimension of $\mathcal{W}$,
- obtain a basis for $\mathcal{W}$ over the reals which is a minimal spanning set extracted from $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots$, and
- express the remaining column vectors as linear combinations of the column vectors in the basis obtained.

(a) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $\mathbf{u}_5 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$.

(b) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -3 \\ 1 \\ -1 \\ 6 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -7 \\ 3 \\ -3 \\ -14 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 5 \end{bmatrix}$, $\mathbf{u}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

(c) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ -3 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 1 \\ 3 \\ -1 \\ -1 \end{bmatrix}$, $\mathbf{u}_5 = \begin{bmatrix} 3 \\ 8 \\ 0 \\ -5 \end{bmatrix}$.

(d) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 2 \\ 4 \\ 3 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 3 \\ 4 \\ 3 \\ 4 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 4 \\ 5 \\ 5 \\ 10 \end{bmatrix}$, $\mathbf{u}_5 = \begin{bmatrix} 7 \\ 8 \\ 11 \\ 13 \end{bmatrix}$.

(e) $\mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 3 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 3 \\ -4 \\ 9 \\ 5 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 2 \\ -3 \\ 8 \\ 7 \end{bmatrix}$, $\mathbf{u}_5 = \begin{bmatrix} 1 \\ 0 \\ 5 \\ 2 \end{bmatrix}$, $\mathbf{u}_6 = \begin{bmatrix} 5 \\ -8 \\ 19 \\ 17 \end{bmatrix}$.

11. Let A be a (3×6)-matrix with real entries, $\mathbf{b} \in \mathbb{R}^3$, and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^6$.

Suppose $\mathbf{u}$ is a solution of $\mathcal{LS}(A, \mathbf{b})$, and $\mathbf{v}$ is a solution of $\mathcal{LS}(A, \mathbf{3b})$.

(a) Write down, if such exists, a scalar multiple of $\mathbf{u}$ which is also a solution of $\mathcal{LS}(A, 3\mathbf{b})$.
Justify your answer.

(b) Write down, if such exists, some $\mathbf{y} \in \mathbb{R}^6$ which simultaneously satisfies $(\sharp_1), (\sharp_2)$:—

$(\sharp_1)$ $\mathbf{y} \in \mathcal{N}(A)$,

$(\sharp_2)$ $\mathbf{y} = \alpha\mathbf{u} + \beta\mathbf{v}$ for some non-zero real numbers α, β.

Give your answer in the form of an appropriate linear combination of $\mathbf{u}, \mathbf{v}$.
Justify your answer.

(c) Suppose $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \in \mathbb{R}^6$, and $\mathcal{N}(A) = \text{Span}(\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\})$.
Fill in the blanks, labelled (I), (II), with appropriate non-zero real numbers to make $(*)$ a true statement:—

$(*)$ *There exist some* $c_1, c_2, c_3 \in \mathbb{R}$ *such that* $\underset{\text{(I)}}{\underline{\quad\quad}}\mathbf{u} + \underset{\text{(II)}}{\underline{\quad\quad}}\mathbf{v} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3$.

(d) Further suppose it is known that $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -1 \\ 4 \\ -5 \\ 5 \\ 3 \\ 3 \end{bmatrix}$, $\mathbf{w}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} \kappa \\ 0 \\ \lambda \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{w}_3 = \begin{bmatrix} -2 \\ 0 \\ 2 \\ 0 \\ 3 \\ 1 \end{bmatrix}$, where κ, λ are some real numbers.

What are the values of κ, λ? Justify your answer.

Remark. You do not need to know what $A, \mathbf{b}$ are, and you are not required to find them.

12. Let $A = \begin{bmatrix} 1 & 1 & 1 & 0 & 2 & 3 & -1 & 1 \\ 3 & 2 & 1 & -2 & 7 & 1 & -2 & -3 \\ 2 & 3 & 4 & -1 & 6 & 5 & -3 & 2 \\ 1 & 2 & 3 & -1 & 4 & 2 & -2 & 1 \\ -1 & -1 & -1 & -2 & 0 & -9 & 2 & -2 \\ 4 & 3 & 2 & -2 & 9 & 4 & -3 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 & 0 & 2 & 3 & -1 & 1 \\ 0 & 1 & 2 & 2 & -1 & 8 & -1 & 6 \\ 0 & 0 & 0 & 1 & -1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Denote the columns of A, from left to right, by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6, \mathbf{a}_7, \mathbf{a}_8$.

Take for granted that A is row-equivalent to B, and that B is a row-echelon form.

(a) Identify the pivot columns in B.

(b) What is the rank of A?

(c) Write down a basis for $\text{Span}(\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6, \mathbf{a}_7, \mathbf{a}_8\})$ which is extracted as a minimal spanning set from amongst the column vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6, \mathbf{a}_7, \mathbf{a}_8$.

(d) i. What is the dimension of $\mathcal{C}(A)$?

ii. Name a basis for $\mathcal{C}(A)$ from amongst the column vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6, \mathbf{a}_7, \mathbf{a}_8$.

iii. Write down the reduced row-echelon form C which is row-equivalent to A.
Hence, or otherwise, express $\mathbf{a}_8$ in terms of the column vectors in the basis for $\mathcal{C}(A)$ that you have named in the previous part.

(e) i. What is the dimension of $\mathcal{N}(A)$?

ii. Name a basis for $\mathcal{N}(A)$.

(f) i. What is the dimension of $\mathcal{R}(A)$?

ii. Name a basis for $\mathcal{R}(A)$ from amongst the columns of C^t, if such exists.

iii. Name a basis for $\mathcal{R}(A)$ from amongst the columns of B^t, if such exists.

iv. What is the dimension of $\mathcal{N}(A^t)$?

13. Let A be a (6×9)-matrix with real entries, whose columns from left to right are denoted by $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6, \mathbf{u}_7, \mathbf{u}_8, \mathbf{u}_9$ respectively.

Let B be a (6×9)-matrix with real entries, given by

$$B = \begin{bmatrix} 1 & 0 & c_{13} & 0 & c_{15} & c_{16} & 0 & 0 & c_{19} \\ 0 & 1 & c_{23} & 0 & c_{25} & c_{26} & 0 & 0 & c_{29} \\ 0 & 0 & 0 & 1 & c_{35} & c_{36} & 0 & 0 & c_{39} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & c_{49} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & c_{59} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

in which the c_{ij}'s are some real numbers.

Suppose A is row-equivalent to B.

(a) Is B a reduced row-echelon form?

- If *yes*, also name all pivot columns of B.
- If *no*, write '*B is not a reduced row-echelon form*'.

(b) i. Name a basis for $\mathcal{C}(A)$ over the reals from amongst the column vectors

$$\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6, \mathbf{u}_7, \mathbf{u}_8, \mathbf{u}_9.$$

ii. What is the dimension of $\mathcal{C}(A)$ over the reals?

(c) i. Name those sets amongst $\mathcal{R}(A^t), \mathcal{C}(A), \mathcal{R}(B), \mathcal{C}(B^t), \mathcal{R}(B^t)$ which are equal to $\mathcal{R}(A)$.

ii. Write down a basis for $\mathcal{R}(A)$ over the reals.

iii. What is the dimension of $\mathcal{R}(A)$ over the reals?

(d) i. Write down an equality which relates the respective dimensions of $\mathcal{N}(A^t)$ and $\mathcal{C}(A^t)$ over the reals.

ii. What is the dimension of $\mathcal{N}(A^t)$ over the reals?

14. Let A, C be (4×6)-matrices respectively given by

$$A = \begin{bmatrix} 1 & a_{12} & 3 & a_{14} & 2 & 2 \\ 0 & a_{22} & 2 & 1 & 2 & 1 \\ -1 & a_{32} & 3 & a_{34} & 2 & 3 \\ 2 & a_{42} & 0 & a_{44} & 5 & -6 \end{bmatrix}, \qquad C = \begin{bmatrix} c_{11} & c_{12} & 1 & c_{14} & 1 & c_{16} \\ c_{21} & c_{22} & 2 & c_{24} & 1 & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ 0 & 0 & 0 & 0 & c_{45} & c_{46} \end{bmatrix}$$

in which the a_{ij}'s, c_{ij}'s are some numbers.

It is known that:—

- the matrix A row-equivalent to C, and
- C is a reduced row-echelon form whose 2-nd column is a pivot column, and whose 6-th column is a free column.

(a) i. Is the 1-st column in C a pivot column? How about the 3-rd column in C? Why?

ii. What are the values of $c_{11}, c_{21}, c_{31}, c_{12}, c_{22}, c_{32}, c_{33}$ and $a_{12}, a_{22}, a_{32}, a_{42}$?

iii. Is the 5-th column in C a pivot column or a free column? How about the 4-th column in C? Why?

iv. What are the values of $c_{14}, c_{24}, c_{34}, c_{35}, c_{45}, c_{16}, c_{26}, c_{36}, c_{46}$, and a_{41}, a_{43}, a_{44}?

(b) What is the dimension of $\mathcal{C}(A)$? What is the dimension of $\mathcal{N}(A)$?

(c) We denote the columns of A, from left to right, by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6$.

For each collection of column vectors below, decide whether it constitutes a bases for $\mathcal{C}(A)$?

i. $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$,

ii. $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$,

iii. $\mathbf{a}_3, \mathbf{a}_4$,

iv. $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4$,

v. $\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_6$.

Justify your answer.

(d) i. What is the dimension of $\mathcal{R}(A)$?

ii. We denote the rows of C, from top to bottom, by $\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, \mathbf{d}_4$.
Name a basis for $\mathcal{R}(A)$ amongst $\mathbf{d}_1{}^t, \mathbf{d}_2{}^t, \mathbf{d}_3{}^t, \mathbf{d}_4{}^t$, if such exists.

(e) Determine whether the statement ($\sharp$) is true or false. Justify your answer.

($\sharp$): *Any two non-trivial solutions of the homogeneous system* $\mathcal{LS}(A^t, \mathbf{0}_6)$ *are scalar multiples of each other.*

15. Let $A = \begin{bmatrix} 1 & 0 & 1 & 1 & 2 & 4 \\ 2 & 1 & -1 & -3 & 4 & 5 \\ -2 & -1 & 1 & 3 & -3 & -4 \end{bmatrix}$, $A' = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 2 \\ 0 & 1 & -3 & -5 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$.

Take for granted that A' is the reduced row-echelon form which is row-equivalent to A.

Let $\mathbf{z}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{z}_2 = \begin{bmatrix} -2 \\ 5 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{z}_3 = \begin{bmatrix} -4 \\ 9 \\ 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

(a) What is the dimension of $\mathcal{N}(A)$?

(b) Do $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ constitute a basis for $\mathcal{N}(A)$? Justify your answer.

16. (a) Let B be a $(p \times q)$-matrix with real entries.

With reference to the definition for the notions of set equality and of column space, prove that the statements ($\sharp_1$), ($\sharp_2$) are logically equivalent:—

($\sharp_1$) *For any* $c \in \mathbb{R}^p$, *the system* $\mathcal{LS}(B, \mathbf{c})$ *is consistent.*

($\sharp_2$) $\mathcal{C}(B) = \mathbb{R}^p$.

Remark. Recall how column space is defined in terms of span, and how consistency of systems can be reformulated in terms of linear combinations.

(b) Let α be a real number, and $A_\alpha = \begin{bmatrix} 1 & 0 & 2 & -3 \\ 2 & 0 & 4 & -6 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 2 & 5 & 5 & \alpha \end{bmatrix}$.

i. For which value(s) of α is it true that $\mathcal{N}(A_\alpha) = \{\mathbf{0}_4\}$? Justify your answer.

ii. Let $\beta, c_1, c_2, c_3, c_4$ be real numbers.
For which value(s) of β is the system $(S_{\beta;c_1,c_2,c_3,c_4})$

$$(S_{\beta;c_1,c_2,c_3,c_4}) \quad \left\{ \begin{array}{rcrcrcrcrcl} x_1 & + & 2x_2 & & & & & + & 2x_5 & = & c_1 \\ & & & & x_3 & + & x_4 & + & 5x_5 & = & c_2 \\ 2x_1 & + & 4x_2 & & & + & x_4 & + & 5x_5 & = & c_3 \\ -3x_1 & - & 6x_2 & + & x_3 & + & 3x_4 & + & \beta x_5 & = & c_4 \end{array} \right.$$

inconsistent for some values of c_1, c_2, c_3, c_4?
Justify your answer. You may apply results which are related to the Rank-nullity Formulae, if relevant and applicable.

17. We denote the rank of an arbitrary matrix with real entries, say, D, by $r(D)$.

Take for granted the result ($\sharp$):—

($\sharp$) *Let* B, C *are square matrices of the same size with real entries. Suppose* $r(B) = s$, $r(C) = t$. *Then the rank of* BC *is at most* $\min(s, t)$.

(a) Apply mathematical induction to prove the statement below:—

Suppose A *is a* $(p \times p)$*-square matrix. Then, for any positive integer* n, *the inequality* $r(A^{n+1}) \leq r(A^n)$.

(b) Name an appropriate (2×2)-square matrix F for which $r(F^n) = r(F)$ for each positive integer n.

(c) Name an appropriate (4×4)-square matrix G for $r(G^4) < r(G^3) < r(G^2) < r(G)$.

18. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^5$.

Suppose none of $\mathbf{u}$, $\mathbf{v}, \mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}$ is the zero column vector.

Prove that the statements below are logically equivalent, with direct reference to the definitions for the notions of set equality and span:—

(1) $\mathbf{u}, \mathbf{v}$ *are non-zero scalar multiples of each other.*

(2) $\text{Span}(\{\mathbf{u}\}) = \text{Span}(\{\mathbf{u}, \mathbf{v}\})$.

(3) $\text{Span}(\{\mathbf{u}\}) = \text{Span}(\{\mathbf{u}+\mathbf{v}\})$.

19. Suppose A is a (7×8)-matrix with real entries, B is an (8×8)-square matrix with real entries, and C is an (8×9)-matrix with real entries.

(a) Show that $\mathcal{N}\big(3A(B^2 - 3B)C\big) = \mathcal{N}\big(A(B^2 - 3B)C\big)$ are equal as sets.

(b) What is the value of $\dim(\mathcal{N}\big(3A(B^2 - 3B)C\big) + \dim(\mathcal{C}\big(2A(B^2 - 3B)C\big))$?

Justify your answer.

20. Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{v} \in \mathbb{R}^9$, and $\mathcal{V} = \text{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\})$, $\mathcal{W} = \text{Span}(\{\mathbf{v}, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\})$.

Suppose the statements (1), (2), (3) hold:—

(1) $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ *are linearly independent over the reals.*

(2) $\mathbf{v}$ *is the linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ *with respect to real scalars* $\alpha_1, \alpha_2, \alpha_3, \alpha_4$.

(3) $\alpha_1 \neq 0$.

(a) Verify that $\mathbf{v}, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are linearly independent are linearly independent.

(b) Verify the statements below:—

i. *For any* $\mathbf{x} \in \mathbb{R}^9$, *if* $\mathbf{x} \in \mathcal{V}$ *then* $\mathbf{x} \in \mathcal{W}$.

ii. *For any* $\mathbf{x} \in \mathbb{R}^9$, *if* $\mathbf{x} \in \mathcal{W}$ *then* $\mathbf{x} \in \mathcal{V}$.

Remark. According to part (b), we may conclude that $\mathcal{V} = \mathcal{W}$ as sets. Overall, the result described by the entire question is an illustration on what is usually known as 'Replacement Theorem' in linear algebra.

21. Let a, b, c be positive real numbers, and k be a real number.

Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} k \\ a \\ a^2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1+a^2 \\ b \\ b^2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 1+a^2 \\ 0 \\ 1+b^2 \\ c \\ c^2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 0 \\ -a \\ 1 \\ b \\ b^2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -b \\ 1 \\ c \\ c^2 \end{bmatrix}.$$

and

$$\mathcal{W} = \text{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}).$$

Suppose ${\mathbf{v}_1}^t\mathbf{v}_2 = 0$.

(a) What is the value of k? Justify your answer.

(b) Show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly independent, with direct reference to the definition of linear dependence/independence.

(c) i. Show that $\mathbf{x}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

ii. Show that $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{x}$.

iii. Hence, or otherwise, show that the statement $(\sharp_1)$ is true, with reference to the definition for set equality:—

$(\sharp_1)$ $\text{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}) = \text{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{x}\})$.

(d) Determine whether the statement $(\sharp_2)$ is true or false. Justify your answer.

($\sharp_2$) $\text{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}) = \text{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{x}, \mathbf{y}\})$.

22. Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5 \in \mathbb{R}^8$, and $\mathcal{V} = \text{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\})$.

    Define $\mathbf{x} = \mathbf{u}_1 - \mathbf{u}_2 + \mathbf{u}_3$, $\mathbf{y} = \mathbf{u}_2 + \mathbf{u}_4 - \mathbf{u}_5$.

    Suppose $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$ are linearly independent over the reals.

    (a) Are $\mathbf{x}, \mathbf{y}, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$ linearly independent over the reals? Justify your answer with direct reference to the definition of *linear independence*.

    (b) Does every linear combination of $\mathbf{x}, \mathbf{y}, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$ belong to $\mathcal{V}$? Justify your answer with direct reference to the definition of *linear combinations*.

    (c) Is every column vector belonging to $\mathcal{V}$ a linear combination of $\mathbf{x}, \mathbf{y}, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$? Justify your answer with direct reference to the definition of *linear combinations*.

    (d) Do $\mathbf{x}, \mathbf{y}, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$ constitute a basis for $\mathcal{V}$? Justify your answer with reference to the definition of *basis*.

23. Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v}_4, \mathbf{v}$ be column vectors belonging to $\mathbb{R}^7$.

    Prove that the statements (1), (2) are logically equivalent, with direct reference to the definitions for the notions of set equality, span, linear combination:—

    (1) *$\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ over the reals.*

    (2) $\text{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{v}\}) = \text{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\})$.

24. Determine whether the statement ($\sharp$) is true. Justify your answer with an appropriate argument with reference to the definitions for the notions of row space, and invertible matrices.

    ($\sharp$) *Let A, B be (6×8)-matrices with real entries, and P, Q are invertible (6×6)-matrices with real entries. Suppose $\mathcal{R}(A) = \mathcal{R}(B)$. Then $\mathcal{R}(PA) = \mathcal{R}(QB)$.*

25. Let A, B be (5×7)-matrices. Suppose $\mathcal{R}(A) = \mathcal{R}(B)$.

    (a) Verify the statements below:—

    i. *Suppose the rank of A is 1. Then A is row-equivalent to B.*
    ii. *Suppose the rank of A is 2. Then A is row-equivalent to B.*

    Remark. What can you say about the reduced row-echelon forms A', B' respectively row-equivalent to A, B? In particular, what can be said of the non-zero rows in A', B'?

    (b) Determine whether the statement ($\sharp$) is true. Justify your answer.

    ($\sharp$) *Suppose the rank of A is 3. Then A is row-equivalent to B.*

26. Determine whether the statement ($\sharp$) is true. Justify your answer with an appropriate argument, with reference to the definitions for the notions of *subspace of $\mathbb{R}^n$*, *basis* and *dimension*.

    ($\sharp$) *Let $\mathcal{U}, \mathcal{V}$ be subspaces of $\mathbb{R}^5$ over the reals, and $\mathcal{W} = \mathcal{U} \cap \mathcal{V}$.*

    Suppose $\dim(\mathcal{U}) = 3$ and $\dim(\mathcal{V}) = 3$.

    Then $\mathcal{W}$ contains a non-zero column vector belonging to $\mathbb{R}^5$.

27. Prove the statement ($\sharp$):

    ($\sharp$) *Let A, B be two (5×5)-square matrices with real entries.*

    Let C be the (5×5)-square matrix defined by $C = A^t A + B^t B$, and D be the (10×5)-matrix defined by

$$D = \left[\begin{array}{c} A \\ \hline B \end{array}\right].$$

    Suppose C is not invertible. Then the rank of D is at most 4.

    Remark. The Rank-nullity Formula may be useful at some point of the argument.

    You may also find the statement ($\natural$) useful at some point:—

($\natural$) *Let* $\mathbf{y} \in \mathbb{R}^n$. *Suppose* $\mathbf{v}^t\mathbf{v} = 0$. *Then* $\mathbf{v} = \mathbf{0}_n$.

(Can you give a proof for ($\natural$)?)

28. Recall the definition for the notion of subspace of a subspace:

Let $\mathcal{V}, \mathcal{W}$ *be a subspace of* $\mathbb{R}^n$ *over the reals.*

We say that $\mathcal{V}$ *is a* **subspace of** $\mathcal{W}$ **over the reals** *if and only if the statement* ($\dagger$) *holds:—*

($\dagger$) *For any* $\mathbf{x} \in \mathbb{R}^n$, *if* $\mathbf{x} \in \mathcal{V}$ *then* $\mathbf{x} \in \mathcal{W}$.

(a) Prove the statement ($\sharp$):—

($\sharp$) *Suppose* A *is an* $(m \times n)$*-matrix with real entries, and* B *is an* $(n \times p)$*-matrix with real entries. Then* $\mathcal{C}(AB)$ *is a subspace of* $\mathcal{C}(A)$.

(b) Dis-prove the statement ($\natural$) by providing a counter-example against it:—

($\natural$) *Suppose* A *is an* (2×2)*-matrix with real entries, and* B *is an* (2×2)*-matrix with real entries. Then* $\mathcal{C}(A)$ *is a subspace of* $\mathcal{C}(AB)$.

Remark. Can you name some appropriate A, B for which $\mathcal{C}(A) = \mathbb{R}^2$ and $\mathcal{C}(B) \neq \mathbb{R}^2$?

(c) For each of the statements below, determine whether it is true or false. Justify your answer.

i. *Suppose* A *is an* $(m \times n)$*-matrix with real entries, and* B *is an* $(n \times p)$*-matrix with real entries. Further suppose* $\mathcal{C}(B) = \mathbb{R}^n$. *Then* $\mathcal{C}(A)$ *is a subspace of* $\mathcal{C}(AB)$.

ii. *Suppose* A *is an* $(m \times n)$*-matrix with real entries, and* B *is an* $(n \times n)$*-matrix with real entries. Further suppose* B *is invertible. Then* $\mathcal{C}(A)$ *is a subspace of* $\mathcal{C}(AB)$.

iii. *Suppose* A *is an* $(m \times n)$*-matrix with real entries, and* B *is an* $(n \times n)$*-matrix with real entries. Further suppose* $\mathcal{C}(A)$ *is a subspace of* $\mathcal{C}(AB)$. *Then* B *is invertible.*

iv. *Suppose* B *is an* $(n \times n)$*-matrix with real entries. Further suppose* $\mathcal{C}(A)$ *is a subspace of* $\mathcal{C}(AB)$ *for every* $(n \times n)$*-square matrix* A. *Then* B *is invertible.*

29. Take for granted the validity of the results ($\star$) below:—

($\star$) *Suppose* G *is an* $(m \times n)$*-matrix with real entries, and* H *is an* $(n \times p)$*-matrix with real entries.*
Then $\mathcal{N}(GH)$ *is a subspace of* $\mathcal{N}(H)$.

(a) Prove the statement ($\sharp$):

($\sharp$) *Suppose* A *is an* $(m \times n)$*-matrix with real entries,* B *is an* $(n \times p)$*-matrix with real entries, and* C *is a* $(p \times q)$*-matrix with real entries.*
Suppose $\mathcal{N}(AB)$ *is a subspace of* $\mathcal{N}(B)$ *over the reals.*
Then $\mathcal{N}(ABC)$ *is a subspace of* $\mathcal{N}(BC)$ *over the reals.*

(b) Hence, or otherwise, deduce the statement ($\sharp\sharp$):

($\sharp\sharp$) *Suppose* A *is an* $(m \times n)$*-matrix with real entries,* B *is an* $(n \times p)$*-matrix with real entries, and* C *is a* $(p \times q)$*-matrix with real entries.*
Suppose $\mathcal{N}(AB) = \mathcal{N}(B)$.
Then $\mathcal{N}(ABC) = \mathcal{N}(BC)$.

(c) Prove the statement ($\natural$):

($\natural$) *Let* E *be a* $(p \times q)$*-matrix with real entries, and* D *be a* $(p \times p)$*-matrix with real entries.*
Suppose there is some positive integer ℓ *such that* $\mathcal{N}(D^\ell) = \mathcal{N}(D^{\ell+1})$.
Then (for the same ℓ*,)* $\mathcal{N}(D^\ell E) = \mathcal{N}(D^{\ell+n}E)$ *for any positive integer* n.

(d) Name some (6×6)-matrix D which satisfies both conditions (1), (2) below:—

(1) $\dim(\mathcal{N}(D)) = 1$, $\dim(\mathcal{N}(D^2)) = 2$, $\dim(\mathcal{N}(D^3)) = 3$, $\dim(\mathcal{N}(D^4)) = 4$.

(2) $\mathcal{N}(D^n) = \mathcal{N}(D^4)$ for each positive integer n greater than 4.

Justify your answer.

30. (a) Let A be a $(p \times q)$-matrix with real entries.

Prove the statements below:—

i. $\mathcal{C}(A) = \left\{ \mathbf{y} \in \mathbb{R}^p \;\middle|\; \text{The system } \mathcal{LS}(A, \mathbf{y}) \text{ is consistent} \right\}$.

ii. *For each integer k between 0 and p, $\dim(\mathcal{C}(A)) = p - k$ if and only if $\dim(\mathcal{N}(A^t)) = k$.*

Remark. Apply the Rank-nullity Formula, where relevant and appropriate.

(b) Determine whether the statement is true or false. Justify your answer with an appropriate argument:—

Let A be a (5×7)-matrix with real entries.

Suppose $\mathcal{LS}(A^t, \mathbf{0}_7)$ has a non-trivial solution, say, $\mathbf{u}$, and $\mathcal{N}(A^t) = Span(\{\mathbf{u}\})$.

then there exists some non-zero column vector $\mathbf{b} \in \mathbb{R}^5$ such that both statements (1), (2) hold:

(1) *For any $\alpha \in \mathbb{R}$, if $\alpha \neq 0$ then $\mathcal{LS}(A, \alpha\mathbf{b})$ is inconsistent.*

(2) *For any $\mathbf{d} \in \mathbb{R}^5$, if $\mathcal{LS}(A, \mathbf{d})$ is inconsistent, then there exists some unique $\beta \in \mathbb{R}$ such that $\mathcal{LS}(A, \mathbf{d} - \beta\mathbf{b})$ is consistent.*