4.7.1 Appendix: Proofs on theoretical results about dimension for general subspaces of $\mathbb{R}^n$.

0. The material in this appendix is supplementary.

1. Here we proceed to prove Theorem (1) and Theorem (5).

   Theorem (1). (Equivalent formulation for the notion of basis.)

   Let $\mathcal{W}$ be a subspace of $\mathbb{R}^q$. Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n \in \mathcal{W}$.

   Denote by (1), (2), (3) the statements below (about $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ and $\mathcal{W}$):

   (1) $\dim(\mathcal{W}) = n$.

   (2) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are linearly independent over the reals.*

   (3) *Every column vector belonging to $\mathcal{W}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ over the reals.*

   Suppose any two statements amongst (1), (2), (3) hold.

   Then all three statements (1), (2), (3) hold, and $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ constitute a basis for $\mathcal{W}$ over the reals.

   Theorem (5). (Inequality for dimensions of 'comparable' subspaces.)

   Let $\mathcal{V}, \mathcal{W}$ be subspaces of $\mathbb{R}^q$ over the reals.

   Suppose $\mathcal{V}$ is a subspace of $\mathcal{W}$ over the reals.

   Then $\dim(\mathcal{V}) \leq \dim(\mathcal{W})$. Moreover, equality holds if and only if $\mathcal{V} = \mathcal{W}$.

2. Theorem (5) will be used in the argument for Theorem (1), and therefore will be proved first.

   The proof of Theorem (5) relies on the result below that we have introduced earlier and labeled Theorem ($\star$) here, and which is built upon the Replacement Theorem.

   Theorem ($\star$).

   Let $\mathcal{W}$ be a n-dimensional subspace of $\mathbb{R}^q$ over the reals. Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k \in \mathcal{W}$.

   Suppose $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ are linearly independent over the reals. Then:—

   (a) *the inequality $k \leq n$ holds, and*

   (b) *there is some basis for $\mathcal{W}$ over the reals constituted by $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$, and some $n-k$ column vectors belonging to $\mathcal{W}$.*

3. **Proof of Theorem (5).**

   Let $\mathcal{V}, \mathcal{W}$ be subspaces of $\mathbb{R}^q$ over the reals.

   Suppose $\mathcal{V}$ is a subspace of $\mathcal{W}$ over the reals. Write $\dim(\mathcal{V}) = k$, and $\dim(\mathcal{W}) = n$.

   Pick some basis for $\mathcal{V}$ over the reals, say, $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$.

   Then by the definition for the notion of subspace of a subspace, each of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ belongs to $\mathcal{W}$.

   By definition for the notion of basis, $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ are linearly independent over the reals.

   So they are k linearly independent column vectors belonging to $\mathcal{W}$.

   Hence, by Theorem ($\star$), $\dim(\mathcal{V}) = k \leq n = \dim(\mathcal{W})$.

   Moreover, there is some basis for $\mathcal{W}$ over the reals constituted by $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$, and some $n-k$ column vectors belonging to $\mathcal{W}$.

   We now proceed to verify that $\dim(\mathcal{V}) = \dim(\mathcal{W})$ if and only if $\mathcal{V} = \mathcal{W}$.:—

   (a) Suppose $\dim(\mathcal{V}) = \dim(\mathcal{W})$. Then $n - k = \dim(\mathcal{W}) - \dim(\mathcal{V}) = 0$.
   So $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ already constitute a basis for $\mathcal{W}$ over the reals.
   It follows that $\mathcal{W} = \mathsf{Span}\ (\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}) = \mathcal{V}$.

   (b) Suppose $\mathcal{V} = \mathcal{W}$. Then $\dim(\mathcal{V}) = \dim(\mathcal{W})$.

4. Now we give an argument for Theorem (1). It is essentially made up of two lemmas.

   Lemma ($\sharp$).

   Let $\mathcal{W}$ be a subspace of $\mathbb{R}^q$. Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n \in \mathcal{W}$.

   Further suppose $\dim(\mathcal{W}) = n$, and $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are linearly independent over the reals.

Then $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ constitute a basis for $\mathcal{W}$ over the reals (and in particular, every column vector belonging to $\mathcal{W}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ over the reals).

Lemma ($\natural$).

Let $\mathcal{W}$ be a subspace of $\mathbb{R}^q$. Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n \in \mathcal{W}$.

Further suppose $\dim(\mathcal{W}) = n$, and every column vector belonging to $\mathcal{W}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ over the reals.

Then $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ constitute a basis for $\mathcal{W}$ over the reals (and, in particular, $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are linearly independent over the reals).

5. **Proof of Lemma ($\sharp$).**

Let $\mathcal{W}$ be a subspace of $\mathbb{R}^q$. Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n \in \mathcal{W}$.

Further suppose $\dim(\mathcal{W}) = n$, and $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are linearly independent over the reals.

Define $\mathcal{V} = \mathrm{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\})$.

Since $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are linearly independent over the reals, they constitute a basis for $\mathcal{V}$ over the reals. Therefore $\dim(V) = n = \dim(\mathcal{W})$.

We verify that $\mathcal{V}$ is a subspace of $\mathcal{W}$ over the reals.:—

- By definition, $\mathcal{V}$ is a subspace of $\mathbb{R}^q$. By assumption, $\mathcal{W}$ is a subspace of $\mathbb{R}^q$.
  [We now verify the statement '*For any $\mathbf{x} \in \mathbb{R}^q$, if $\mathbf{x} \in \mathcal{V}$ then $\mathbf{x} \in \mathcal{W}$*'.]
  Pick any $\mathbf{x} \in \mathbb{R}^q$. Suppose $\mathbf{x} \in \mathcal{V}$. By definition of span, $\mathbf{x}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.
  Then, since $\mathcal{W}$ is a subspace of $\mathbb{R}^q$ over the reals and $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ belong to $\mathcal{W}$, $\mathbf{x}$ belongs to $\mathcal{W}$.

Now by Theorem (5), since $\mathcal{V}$ is a subspace of $\mathcal{W}$ and $\dim(\mathcal{V}) = \dim(\mathcal{W})$, we have $\mathcal{V} = \mathcal{W}$.

Hence $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ constitute a basis for $\mathcal{W}$ over the reals, (and in particular, every column vector belonging to $\mathcal{W}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$).

6. **Proof of Lemma ($\natural$).**

Let $\mathcal{W}$ be a subspace of $\mathbb{R}^q$. Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n \in \mathcal{W}$.

Further suppose $\dim(\mathcal{W}) = n$, and every column vector belonging to $\mathcal{W}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ over the reals.

By assumption, $\mathcal{W} = \mathrm{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\})$. We may extract from $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$, a minimal spanning set for $\mathcal{W}$, which consists of r column vectors $\mathbf{u}_{d_1}, \mathbf{u}_{d_2}, \cdots, \mathbf{u}_{d_r}$. Here r is the rank of the matrix $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_n\ ]$, and the pivot columns in the reduced row-echelon form U' which is row-equivalent to U are the d_1-th, d_2-th, ..., d_r-th columns of U'.

These r column vectors $\mathbf{u}_{d_1}, \mathbf{u}_{d_2}, \cdots, \mathbf{u}_{d_r}$ constitute a basis for $\mathcal{W}$. Therefore $\dim(\mathcal{W}) = r$.

Recall that by assumption, $\dim(\mathcal{W}) = n$. Then $n = r$. Therefore $\mathbf{u}_{d_1}, \mathbf{u}_{d_2}, \cdots, \mathbf{u}_{d_r}$ are all of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

It follows that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ constitute a basis for $\mathcal{W}$, (and, in particular, $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are linearly independent over the reals).

7. We now complete the proof of Theorem (1).

Proof of Theorem (1).

Let $\mathcal{W}$ be a subspace of $\mathbb{R}^q$. Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n \in \mathcal{W}$.

(a) Suppose $\dim(\mathcal{W}) = n$ and $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are linearly independent over the reals.
Then, by Lemma ($\sharp$), $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ constitute a basis for $\mathcal{W}$ over the reals (and, in particular, every column vector belonging to $\mathcal{W}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ over the reals).

(b) Suppose $\dim(\mathcal{W}) = n$ and every column vector belonging to $\mathcal{W}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ over the reals.
Then, by Lemma ($\natural$), $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ constitute a basis for $\mathcal{W}$ over the reals (and, in particular, $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are linearly independent over the reals).

(c) Suppose that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are linearly independent over the reals, and every column vector belonging to $\mathcal{W}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ over the reals.
Then, by definition, $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ constitute a basis for $\mathcal{W}$, and $\dim(\mathcal{W}) = n$.