4.7 Basis and dimension, null space, and the notion of subspace of a subspace.

0. *Assumed background.*
   - Whatever has been covered in Topics 1-3, especially:—
     - ∗ 2.2 *Row-echelon forms and reduced row-echelon forms.*
     - ∗ 2.3 *Existence of reduced row-echelon form row-equivalent to given matrix, (and the uniqueness question).*
     - ∗ 2.4 *Solving systems of linear equations.*
   - 4.4 *Basis for subspaces of column vectors.*
   - 4.5 *Dimension for subspaces of column vectors.*
   - 4.6 *Span, column space, row space, and minimal spanning set.*

   Abstract. We introduce:—
   - a theoretical result which re-formulates our earlier understanding about homogeneous systems of linear equations in terms of null space, span, basis and dimension, and informs us about the dimension of the null space of an arbitrary matrix,
   - the notion of subspace of a subspace of $\mathbb{R}^n$,
   - some theoretical results about dimension for general subspaces of $\mathbb{R}^n$, one of them about re-formulation for the notion of basis, and another about inequality between a subspace of $\mathbb{R}^n$ and any one of its subspace.

   In the *appendix*, the proofs for those theoretical results about dimension for general subspaces of $\mathbb{R}^n$ are provided.

1. Below is what we know from the definitions for the notions of basis and dimension for a general subspace of $\mathbb{R}^q$ which is not the zero subspace:—

   Let $\mathcal{W}$ be a subspace of $\mathbb{R}^q$. Suppose $\mathcal{W} \neq \{\mathbf{0}_q\}$.

   Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n \in \mathcal{W}$. Then $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ constitute a **basis** *for $\mathcal{W}$ over the reals if and only if the statements* (BL), (BS) *hold:—*

   (BL) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are linearly independent over the reals.*

   (BS) *Every column vector belonging to $\mathcal{W}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ over the reals.*

   The number of column vectors in the collection $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$, namely n, is called the **dimension** *of $\mathcal{W}$ over the reals.*

2. A big problem with this definition appears when we try to use it in checking whether a collection of column vectors indeed constitutes a basis for a given subspace of $\mathbb{R}^q$ over the reals: a lot of computation has to be done in order to see whether the statement (BS) holds for such a collection of column vectors, because it is essentially about a set equality between the given subspace and the span of those given column vectors.

   We may wonder:—

   Question (1). Is there some equivalent formulation for the notion of basis that will help us do computations in practice?

   An answer to this question is provided by the theoretical result below, whose proof will be provided in the *appendix.*

3. **Theorem (1). (Equivalent formulation for the notion of basis.)**

   Let $\mathcal{W}$ be a subspace of $\mathbb{R}^q$. Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n \in \mathcal{W}$.

   Denote by (1), (2), (3) *the statements below (about $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ and $\mathcal{W}$):*

   (1) $\dim(\mathcal{W}) = n$.

   (2) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are linearly independent over the reals.*

   (3) *Every column vector belonging to $\mathcal{W}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ over the reals.*

   Suppose any two statements amongst (1), (2), (3) *hold.*

   Then all three statements (1), (2), (3) *hold, and $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ constitute a basis for $\mathcal{W}$ over the reals.*

4. **Comment on the content of Theorem (1).** The significance of Theorem (1) lies in the fact that under the assumption '$\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n \in \mathcal{W}$', once it is known that $\dim(\mathcal{W}) = n$, we can immediately confirm whether $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ constitute a basis for $\mathcal{W}$ by only answering one, instead of both, of questions:
   - whether $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are linearly independent over the reals, or

- whether $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ span the whole of $\mathcal{W}$ over the reals.

This will save a lot of effort in calculations.

5. We are going to give an application of Theorem (1) in a concrete situation.

   We prepare our way by stating a result about null space.

   Theorem (2). (Dimension of null space of an arbitrary matrix.)

   Let A be an $(p \times q)$-matrix with real entries.

   Suppose that the rank of A is r.

   Then the dimension of $\mathcal{N}(A)$ over the reals is $q - r$.

   Remark on terminology. The number $\dim(\mathcal{N}(A))$ is very often referred to as the **nullity of** A.

6. The proof of Theorem (2) is no more than a 're-packaging', in terms of null space, span, basis, and dimension, of what we have learnt about full descriptions of all solutions of homogeneous systems earlier on. It is outlined below.

   Outline of proof of Theorem (2).

   Suppose A is a $(p \times q)$-matrix with real entries, of rank r, and A' is the reduced row-echelon form which is row-equivalent to A. We are going to display a basis for $\mathcal{N}(A)$ with $q - r$ column vectors, using the entries of the free columns in A'.

   (a) Suppose that:—

   - the pivot columns of A', from left to right, are the d_1-th, d_2-th, ..., d_r-th columns,
   - the free columns of A', from left to right, are the f_1-th, f_2-th, ..., f_{q-r}-th columns, and
   - for each $\ell = 1, 2, \cdots, q - r$, and for each $k = 1, 2, \cdots, r$, the k-th entry of the f_ℓ-th column is $\alpha_{k\ell}$.

   For each $\ell = 1, 2, \cdots, q - r$, define $\mathbf{h}_\ell$ to be the column vector with q entries whose i-th entry $h_{i\ell}$ is given by

   $$h_{i\ell} = \begin{cases} -\alpha_{j\ell} & \text{if } i = d_j \text{ for some } j = 1, 2, \cdots, r \\ 1 & \text{if } i = f_\ell \\ 0 & \text{otherwise} \end{cases}$$

   (b) Then the statements (L) and (S) hold:

   (L) $\mathbf{h}_1, \mathbf{h}_2, \cdots, \mathbf{h}_{q-r}$ are linearly independent over the reals, and

   (S) for any $\mathbf{t} \in \mathbb{R}^q$, $\mathbf{t}$ is a solution of the homogeneous system $\mathcal{LS}(A, \mathbf{0}_p)$ if and only if $\mathbf{t}$ is a linear combination of $\mathbf{h}_1, \mathbf{h}_2, \cdots, \mathbf{h}_{q-r}$.

   (c) The statement (S) is a re-formulation of the equality

   $$\mathcal{N}(A) = \mathrm{Span}(\{\mathbf{h}_1, \mathbf{h}_2, \cdots, \mathbf{h}_{q-r}\}).$$

   (d) Because of the validity of (L) and (S), we may conclude that $\mathbf{h}_1, \mathbf{h}_2, \cdots, \mathbf{h}_{q-r}$ constitute a basis for the null space of $\mathcal{N}(A)$.

7. We are now ready to give an application of Theorem (1), in answering the question below:—

   Question (2). *Suppose A is a $(p \times q)$-matrix with real entries, and suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are n column vectors with q real entries.*

   How to systematically and methodically determine whether $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ constitute a basis for the null space $\mathcal{N}(A)$ of A, without having to verify $\mathcal{N}(A) = \mathrm{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\})$?

   An answer to Question (2) is in the form of Theorem (3) and its associated algorithm below. As can be seen, we 'favour' the use of the notion of linear independence.

8. **Theorem (3). (Corollary to Theorem (1).)**

   Suppose A is a $(p \times q)$-matrix with real entries, with rank r, and suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n \in \mathbb{R}^q$.

   Then $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ constitute a basis for $\mathcal{N}(A)$ over the reals if and only if all of (a), (b), (c) *below hold simultaneously:—*

   (a) $A\mathbf{u}_j = \mathbf{0}_p$ *for each* $j = 1, 2, \cdots, n$.

   (b) $n = q - r$.

   (c) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are linearly independent over the reals.*

9. **Algorithm associated with Theorem (3).**

Suppose A is a $(p \times q)$-matrix with real entries, and suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are n column vectors with q real entries.

We proceed as described below to determine whether $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ constitute a basis for $\mathcal{N}(A)$:—

Step (1). Ask:—

Is it true that $A\mathbf{u}_j = \mathbf{0}_p$ *for each* $j = 1, 2, \cdots, q$*?*

- If *no*, conclude that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ do not constitute a basis for $\mathcal{N}(A)$ over the reals.
- If *yes*, go to Step (2).

Step (2). Obtain the reduced row-echelon form A' which is row-equivalent to A, and by inspecting A', read off the rank r of A. Conclude that $\dim(\mathcal{N}(A)) = q - r$, and further ask:—

Is it true that $n = q - r$*?*

- If *no*, conclude that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ do not constitute a basis for $\mathcal{N}(A)$ over the reals.
- If *yes*, go to Step (3).

Step (3). Form the matrix $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_n\ ]$, and obtain the reduced row-echelon form U' which is row-equivalent to U. Ask:—

Is every column of U' *a pivot column?*

- If *no*, conclude that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are linearly dependent over the reals, and further conclude that they do not constitute a basis for $\mathcal{N}(A)$ over the reals.
- If *yes*, conclude that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are linearly independent over the reals, and further conclude that they constitute a basis for $\mathcal{N}(A)$ over the reals.

10. **Example (1). (Illustration of the algorithm for determining whether some given column vectors constitute a basis for the null space of a given matrix.)**

(a) *Let* $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & -1 \\ 3 & 1 & 5 & -7 \end{bmatrix}$, *and* $\mathcal{W} = \mathcal{N}(A)$. *Let* $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 5 \\ -3 \\ -1 \\ 1 \end{bmatrix}$.

We check whether $\mathbf{u}_1, \mathbf{u}_2$ *constitute a basis for* $\mathcal{W}$.

i. We have $A\mathbf{u}_1 = \mathbf{0}_3$ and $A\mathbf{u}_2 = \mathbf{0}_3$. Then $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{W}$.

ii. We obtain the reduced row-echelon form A' which is row-equivalent to A:

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & -1 \\ 3 & 1 & 5 & -7 \end{bmatrix} \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A'$$

There are two pivot columns in A'. Then the rank of A is 2.
Therefore $\dim(\mathcal{W}) = 4 - 2 = 2$.
(Recall that there are altogether 2 column vectors amongst $\mathbf{u}_1, \mathbf{u}_2$.)

iii. Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2\ ]$. We obtain the reduced row-echelon form U' which is row-equivalent to U:

$$U = \begin{bmatrix} 1 & 5 \\ -1 & -3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = U'$$

Every column in U' is a pivot column. Then $\mathbf{u}_1, \mathbf{u}_2$ are linearly independent over the reals.
Hence $\mathbf{u}_1, \mathbf{u}_2$ constitute a basis for $\mathcal{W}$.

(b) *Let* $A = \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & -1 & 3 \\ 3 & 1 & 5 & -7 & 1 \end{bmatrix}$, *and* $\mathcal{W} = \mathcal{N}(A)$. *Let* $\mathbf{u}_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -1 \\ 8 \\ -2 \\ -1 \\ -2 \end{bmatrix}$.

We check whether $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ *constitute a basis for* $\mathcal{W}$.

i. We have $A\mathbf{u}_1 = \mathbf{0}_3$, $A\mathbf{u}_2 = \mathbf{0}_3$ and $A\mathbf{u}_3 = \mathbf{0}_3$. Then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathcal{W}$.

ii. We obtain the reduced row-echelon form A' which is row-equivalent to A:

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & -1 & 3 \\ 3 & 1 & 5 & -7 & 1 \end{bmatrix} \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & 0 & 2 & -3 & -1 \\ 0 & 1 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = A'$$

There are two pivot columns in A'. Then the rank of A is 2.
Therefore $\dim(\mathcal{W}) = 5 - 2 = 3$.
(Recall that there are altogether 3 column vectors amongst $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.)

iii. Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3\ ]$. We obtain the reduced row-echelon form U' which is row-equivalent to U:

$$U = \begin{bmatrix} 2 & 2 & -1 \\ -5 & 2 & 8 \\ 1 & 0 & -2 \\ 1 & 1 & -1 \\ 1 & -1 & -2 \end{bmatrix} \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U'$$

Every column in U' is a pivot column. Then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly independent over the reals.
Hence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ constitute a basis for $\mathcal{W}$.

(c) *Let* $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & -1 \\ 3 & 1 & 5 & -7 \end{bmatrix}$*, and* $\mathcal{W} = \mathcal{N}(A)$*. Let* $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

We check whether $\mathbf{u}_1, \mathbf{u}_2$ *constitute a basis for* $\mathcal{W}$.

Note that $A\mathbf{u}_1 = \begin{bmatrix} 0 \\ 6 \\ 30 \end{bmatrix} \neq \mathbf{0}_3$. Then $\mathbf{u}_1 \notin \mathcal{W}$.

Hence $\mathbf{u}_1, \mathbf{u}_2$ do not constitute a basis for $\mathcal{W}$ over the reals.

(d) *Let* $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & -1 \\ 3 & 1 & 5 & -7 \end{bmatrix}$*, and* $\mathcal{W} = \mathcal{N}(A)$*. Let* $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$.

We check whether $\mathbf{u}_1, \mathbf{u}_2$ *constitute a basis for* $\mathcal{W}$.

i. Note that $A\mathbf{u}_1 = \mathbf{0}_3$.
Then $\mathbf{u}_1 \in \mathcal{W}$.

ii. We obtain the reduced row-echelon form A' which is row-equivalent to A:

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & -1 \\ 3 & 1 & 5 & -7 \end{bmatrix} \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A'$$

There are two pivot columns in A'. Then the rank of A is 2.
Therefore $\dim(\mathcal{W}) = 4 - 2 = 2$.
Note that there is only one column vectors amongst $\mathbf{u}_1$. (However, every basis for $\mathcal{W}$ must consist of two column vectors.) Hence $\mathbf{u}_1$ does not constitute a basis for $\mathcal{W}$ over the reals.

(e) *Let* $A = \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & -1 & 3 \\ 3 & 1 & 5 & -7 & 1 \end{bmatrix}$*, and* $\mathcal{W} = \mathcal{N}(A)$*. Let* $\mathbf{u}_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 0 \\ -7 \\ 1 \\ 0 \\ 2 \end{bmatrix}$.

We check whether $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ *constitute a basis for* $\mathcal{W}$.

i. We have $A\mathbf{u}_1 = \mathbf{0}_5$, $A\mathbf{u}_2 = \mathbf{0}_5$ and $A\mathbf{u}_3 = \mathbf{0}_5$. Then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathcal{W}$.

ii. We obtain the reduced row-echelon form A' which is row-equivalent to A:

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & -1 & 3 \\ 3 & 1 & 5 & -7 & 1 \end{bmatrix} \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & 0 & 2 & -3 & -1 \\ 0 & 1 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = A'$$

There are two pivot columns in A'. Then the rank of A is 2.
Therefore $\dim(\mathcal{W}) = 5 - 2 = 3$.
(Recall that there are altogether 3 column vectors amongst $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.)

iii. Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3\ ]$. We obtain the reduced row-echelon form U' which is row-equivalent to U:

$$U = \begin{bmatrix} 2 & 2 & 0 \\ -5 & 2 & -7 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix} \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U'$$

Not every column in U' is a pivot column. Then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly dependent over the reals. Hence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ do not constitute a basis for $\mathcal{W}$.

11. Serving as a pillar of support for Theorem (1) is an important theoretical result about dimension, which we are going to introduce immediately after the definition for the notion of *subspace of a subspace*, with which we formulate the result concerned.

    Definition. (Subspace of a subspace.)

    Let $\mathcal{V}, \mathcal{W}$ be subspaces of $\mathbb{R}^q$ over the reals.

    We say $\mathcal{V}$ is a **subspace of $\mathcal{W}$ over the reals** *if and only if the statement (†) holds:*

    (†) *For any $\mathbf{x} \in \mathbb{R}^q$, if $\mathbf{x} \in \mathcal{V}$ then $\mathbf{x} \in \mathcal{W}$.*

    Remark. In plain words, the statement (†) reads:

    '*every column vector which belongs to $\mathcal{V}$ will automatically belong to $\mathcal{W}$ as well*'.

    We are interested in such a pair of 'comparable' subspaces of $\mathbb{R}^q$ over the real.

    Further remark. As can be seen in Example (2) and Example (3), we should not expect any two arbitrary subspaces of $\mathbb{R}^q$ over the reals to be 'comparable' in the sense of one of them being a subspace of the other.

12. The relation between the notions of set equality and subspace of subspace follows immediately from their respective definitions and logic.

    Lemma (4).

    Suppose $\mathcal{V}, \mathcal{W}$ are subspaces of $\mathbb{R}^q$ over the reals. Then the statements below are logically equivalent:—

    (1) $\mathcal{V} = \mathcal{W}$.

    (2) *$\mathcal{V}$ is a subspace of $\mathcal{W}$ over the reals, and $\mathcal{W}$ is a subspace of $\mathcal{V}$ over the reals.*

13. We are now ready to state the important result about dimension which is a pillar of support for Theorem (1). Its proof is built upon the Replacement Theorem, and is provided in the *appendix*.

    Theorem (5). (Inequality for dimensions of 'comparable' subspaces.)

    Let $\mathcal{V}, \mathcal{W}$ be subspaces of $\mathbb{R}^q$ over the reals.

    Suppose $\mathcal{V}$ is a subspace of $\mathcal{W}$ over the reals.

    Then $\dim(\mathcal{V}) \leq \dim(\mathcal{W})$.

    Moreover, equality holds if and only if $\mathcal{V} = \mathcal{W}$.

    Remark. The significance of Theorem (5) is that, under the assumption of $\mathcal{V}$ being 'part of' $\mathcal{W}$ (in the sense that the former is a subspace of the latter over the reals), whether they are the 'same' is entirely decided by whether their respective dimensions (which are just numbers) are equal to each other.

14. **Example (2). (Illustration on the definition for the notion of subspace of a subspace through null spaces of matrices.)**

    Let $\mathbf{a}_1 = [\ 1 \quad 2 \quad 2 \quad 4\ ]$, $\mathbf{a}_2 = [\ 1 \quad 3 \quad 3 \quad 5\ ]$, $\mathbf{a}_3 = [\ 2 \quad 6 \quad 5 \quad 6\ ]$, *and*

    $$A = \begin{bmatrix} \mathbf{a}_1 \\ \hline \mathbf{a}_2 \\ \hline \mathbf{a}_3 \end{bmatrix}, B = \begin{bmatrix} \mathbf{a}_1 \\ \hline \mathbf{a}_2 \end{bmatrix}, C = \begin{bmatrix} \mathbf{a}_2 \\ \hline \mathbf{a}_3 \end{bmatrix}, D = [\ \mathbf{a}_2\ ].$$

    (a) Without solving any equation, we know that:—

    i. $\mathcal{N}(A)$ is a subspace of $\mathcal{N}(B)$.
    Reason *in plain words*: Every solution of $\mathcal{LS}(A, \mathbf{0}_3)$ will automatically be a solution of $\mathcal{LS}(B, \mathbf{0}_2)$.

    ii. $\mathcal{N}(A)$ is a subspace of $\mathcal{N}(C)$.
    Reason *in plain words*: Every solution of $\mathcal{LS}(A, \mathbf{0}_3)$ will automatically be a solution of $\mathcal{LS}(C, \mathbf{0}_2)$.

    iii. $\mathcal{N}(A)$ is a subspace of $\mathcal{N}(D)$.
    Reason *in plain words*: Every solution of $\mathcal{LS}(A, \mathbf{0}_3)$ will automatically be a solution of $\mathcal{LS}(D, \mathbf{0}_1)$.

    iv. $\mathcal{N}(B)$ is a subspace of $\mathcal{N}(D)$.
    Reason *in plain words*: Every solution of $\mathcal{LS}(B, \mathbf{0}_2)$ will automatically be a solution of $\mathcal{LS}(D, \mathbf{0}_1)$.

    v. $\mathcal{N}(C)$ is a subspace of $\mathcal{N}(D)$.
    Reason *in plain words*: Every solution of $\mathcal{LS}(C, \mathbf{0}_2)$ will automatically be a solution of $\mathcal{LS}(D, \mathbf{0}_1)$.

(b) In fact, the reduced row-echelon forms A', B', C', D' which are respectively row-equivalent to A, B, C, D are given by

$$A' = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix}, B' = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}, C' = \begin{bmatrix} 1 & 3 & 0 & -7 \\ 0 & 0 & 1 & 4 \end{bmatrix}, D' = D = \begin{bmatrix} 1 & 3 & 3 & 5 \end{bmatrix}.$$

Note that the respective dimensions of $\mathcal{N}(A), \mathcal{N}(B), \mathcal{N}(C), \mathcal{N}(D)$ are $1, 2, 2, 3$.
So $\dim(\mathcal{N}(A)) \leq \dim(\mathcal{N}(B))$, $\dim(\mathcal{N}(A)) \leq \dim(\mathcal{N}(C))$, $\dim(\mathcal{N}(A)) \leq \dim(\mathcal{N}(D))$, $\dim(\mathcal{N}(B)) \leq \dim(\mathcal{N}(D))$, $\dim(\mathcal{N}(C)) \leq \dim(\mathcal{N}(D))$, as expected.

(c) By inspecting the matrices B', C', we see that:—

i. $\mathcal{N}(B)$ is not a subspace of $\mathcal{N}(C)$.
Reason *in plain words*: $\begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ is a solution of $\mathcal{LS}(B, \mathbf{0}_2)$, but not a solution of $\mathcal{LS}(C, \mathbf{0}_2)$.

ii. $\mathcal{N}(C)$ is not a subspace of $\mathcal{N}(B)$.
Reason *in plain words*: $\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is a solution of $\mathcal{LS}(C, \mathbf{0}_2)$, but not a solution of $\mathcal{LS}(B, \mathbf{0}_2)$.

It should be noted that '$\dim(\mathcal{N}(B)) = \dim(\mathcal{N}(C))$' does not necessarily imply '$\mathcal{N}(B) = \mathcal{N}(C)$': right in the first place, neither of $\mathcal{N}(B), \mathcal{N}(C)$ is a subspace of the other.

15. Much of Example (2) is an illustration of a general result about null space.

Theorem (6).

Let A be a $(p \times q)$-matrix with real entries. Suppose $\widehat{A}$ is a matrix with q columns whose rows are amongst the rows of A.

Then $\mathcal{N}(A)$ is a subspace of $\mathcal{N}\left(\widehat{A}\right)$ over the reals.

Moreover, $\dim(\mathcal{N}(A)) \leq \dim\left(\mathcal{N}\left(\widehat{A}\right)\right)$. *(So the nullity of A is no greater than the nullity of $\widehat{A}$.)*

Remark. The proof is left as an exercise, as an application of Theorem (5). The argument hinges on the observation (which needs to be justified) that given that the rows of A, from top to bottom, are $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_p$, every solution of $\mathcal{LS}(A, \mathbf{0}_p)$ is a solution of each $\mathcal{LS}(\mathbf{a}_k, \mathbf{0}_1)$.

Further remark. Here we refrain from saying anything about the necessary and sufficient condition for the equality '$\dim(\mathcal{N}(A)) \leq \dim\left(\mathcal{N}\left(\widehat{A}\right)\right)$' to hold. This matter will be handled after we have learnt the Rank-nullity Formula.

16. **Example (3). (Illustration on the definition for the notion of subspace of a subspace through spans.)**

Let $\mathbf{x}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \\ 2 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix}$, $\mathbf{x}_4 = \begin{bmatrix} -6 \\ 7 \\ -1 \\ 1 \\ 1 \end{bmatrix}$, *and*

$$\mathcal{T} = \mathsf{Span}\ (\{\mathbf{x}_1, \mathbf{x}_2\}), \mathcal{U} = \mathsf{Span}\ (\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}), \mathcal{V} = \mathsf{Span}\ (\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4\}), \mathcal{W} = \mathsf{Span}\ (\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}).$$

(a) According to the definition of spans and linear combinations, we know that:—

i. $\mathcal{T}$ is a subspace of $\mathcal{U}$ over the reals.
Reason *in plain words*: Every linear combination of $\mathbf{x}_1, \mathbf{x}_2$ will automatically be a $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$.

ii. $\mathcal{T}$ is a subspace of $\mathcal{V}$ over the reals.
Reason *in plain words*: Every linear combination of $\mathbf{x}_1, \mathbf{x}_2$ will automatically be a $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4$.

iii. $\mathcal{T}$ is a subspace of $\mathcal{W}$ over the reals.
Reason *in plain words*: Every linear combination of $\mathbf{x}_1, \mathbf{x}_2$ will automatically be a $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$.

iv. $\mathcal{U}$ is a subspace of $\mathcal{W}$ over the reals.
Reason *in plain words*: Every linear combination of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ will automatically be a $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$.

v. $\mathcal{V}$ is a subspace of $\mathcal{W}$ over the reals.
Reason *in plain words*: Every linear combination of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4$ will automatically be a $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$.

(b) Write

$$A = [\ \mathbf{x}_1 \mid \mathbf{x}_2\ ], B = [\ \mathbf{x}_1 \mid \mathbf{x}_2 \mid \mathbf{x}_3\ ], C = [\ \mathbf{x}_1 \mid \mathbf{x}_2 \mid \mathbf{x}_4\ ], D = [\ \mathbf{x}_1 \mid \mathbf{x}_2 \mid \mathbf{x}_3 \mid \mathbf{x}_4\ ].$$

The reduced row-echelon forms A', B', C', D' which are respectively row-equivalent to A, B, C, D are given by

$$A' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, B' = C' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, D' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that the respective dimensions of $\mathcal{T}, \mathcal{U}, \mathcal{V}, \mathcal{W}$ are $2, 3, 3, 4$.
So $\dim(\mathcal{T}) \leq \dim(\mathcal{U})$, $\dim(\mathcal{T}) \leq \dim(\mathcal{V})$, $\dim(\mathcal{T}) \leq \dim(\mathcal{W})$, $\dim(\mathcal{U}) \leq \dim(\mathcal{W})$, $\dim(\mathcal{V}) \leq \dim(\mathcal{W})$, as expected.

(c) By inspecting D', we see that that:—

i. $\mathcal{U}$ is not a subspace of $\mathcal{V}$ over the reals.
Reason *in plain words*: $\mathbf{x}_3$ is a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, but not a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4$.

ii. $\mathcal{V}$ is not a subspace of $\mathcal{U}$ over the reals.
Reason *in plain words*: $\mathbf{x}_4$ is a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4$, but not a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$.

It should be noted that '$\dim(\mathcal{U}) = \dim(\mathcal{V})$' does not necessarily imply '$\mathcal{U} = \mathcal{V}$': right in the first place, neither of $\mathcal{U}, \mathcal{V}$ is a subspace of the other.

17. Example (3) is an illustration for a general result about span.

Theorem (7).

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q \in \mathbb{R}^p$, *and* $\mathcal{U} = \mathrm{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\})$.

Suppose $\widetilde{\mathcal{U}}$ *is the span of some amongst* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$, *say,* $\mathbf{u}_{j_1}, \mathbf{u}_{j_2}, \cdots, \mathbf{u}_{j_\ell}$.

Then $\widetilde{\mathcal{U}}$ *is a subspace of* $\mathcal{U}$ *over the reals.*

Moreover, $\dim\left(\widetilde{\mathcal{U}}\right) \leq \dim(\mathcal{U})$.

Equality holds if and only if $\mathbf{u}_k$ *is a linear combination of* $\mathbf{u}_{j_1}, \mathbf{u}_{j_2}, \cdots, \mathbf{u}_{j_\ell}$ *for each* $k = 1, 2, \cdots, q$.

Remark. The proof is left as an exercise on the application of Theorem (5). The argument relies on the definition for the notions of linear combination and span.

18. We can re-formulate Theorem (7) in terms of column space.

Theorem (8). (Corollary to Theorem (7).)

Let G *be a* $(p \times q)$*-matrix with real entries.*

Suppose $\widetilde{G}$ *is a matrix with* p *rows whose columns are some amongst those of* G.

Then $\mathcal{C}\left(\widetilde{G}\right)$ *is a subspace of* $\mathcal{C}(G)$ *over the reals.*

Moreover, $\dim\left(\mathcal{C}\left(\widetilde{G}\right)\right) \leq \dim(\mathcal{C}(G))$. *(So the rank of* $\widetilde{G}$ *is no greater than the rank of* G.*)*

Equality holds if and only if each column of G *is a linear combination of the columns of* $\widetilde{G}$.

19. The result below follows from Theorem (8) and the linkage between row space and column space in their definitions.

Theorem (9). (Corollary to Theorem (7).)

Let H *be an* $(m \times n)$*-matrix with real entries.*

Suppose $\widetilde{H}$ *is a matrix with* n *columns whose rows are amongst those of* H.

Then $\mathcal{R}\left(\widetilde{H}\right)$ *is a subspace of* $\mathcal{R}(H)$ *over the reals.*

Moreover, $\dim\left(\mathcal{R}\left(\widetilde{H}\right)\right) \leq \dim(\mathcal{R}(H))$. *(So the rank of* $\widetilde{H}$ *is no greater than the rank of* H.*)*

Equality holds if and only if each row of H *is a linear combination of the rows of* $\widetilde{H}$.