4.6 Span, column space, row space, and minimal spanning set.

0. *Assumed background.*

   - Whatever has been covered in Topics 1-3, especially:—
     - ∗ 1.8 *Row operations and matrix multiplication.*
     - ∗ 2.2 *Row-echelon forms and reduced row-echelon forms.*
     - ∗ 2.3 *Existence of reduced row-echelon form row-equivalent to given matrix, (and the uniqueness question).*
     - ∗ 3.4 *Row-equivalence from the point of view of invertible matrices.*
   - 4.2 *Set equality (for sets of matrices and sets of vectors).*
   - 4.3 *Subspaces of column vectors.*
   - 4.4 *Basis for subspaces of column vectors.*
   - 4.5 *Dimension for subspaces of column vectors.*

   Abstract. We introduce:—

   - a theoretical result that informs us on how to determine the dimension of the span of an arbitrarily given collection of (finitely many) column vectors with n real entries, and how to obtain a basis for such a subspace of $\mathbb{R}^n$ which is also a 'minimal spanning set' extracted from the given collection of column vectors, alongside an accompanying 'algorithm',
   - the notions of column space, row space, column rank, row rank for an arbitrarily given matrix,
   - a theoretical result which informs us that the column rank, the row rank, and the rank of an arbitrarily given matrix are equal to each other.

   In the *appendix*, we provide the proofs of the theoretical results that we are going to state.

1. Recall the result below, about basis and dimension for span of column vectors:—

   Theorem ($\star$).

   Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q \in \mathbb{R}^n$.

   Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *are linearly independent over the reals.*

   Then:—

   (a) *the inequality* $q \leq n$ *holds, and*

   (b) $\text{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\})$ *is a* q*-dimensional subspace of* $\mathbb{R}^n$ *over the reals, with a basis over the reals constituted by* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.

2. Given that we also talk about spans of not necessarily linearly independent column vectors, Theorem ($\star$) motivates the question below:—

   Question (1).

   What can we say when the assumption

   $$\text{`}\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q \textit{ are linearly independent'}$$

   in the statement of Theorem ($\star$) *has been 'dropped'? How then to determine a basis and (thus) determine the dimension of* $\text{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\})$*?*

3. An answer to Question (1) is hinted by the result below, which has been established earlier:—

   Theorem ($\star\star$).

   Suppose $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r, \mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_s \in \mathbb{R}^n$. *Then the statements below are logically equivalent:—*

   (1) *Each of* $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_s$ *is a linear combination of* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r$ *over the reals.*

   (2) $\text{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r, \mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_s\}) = \text{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r\})$.

4. **Answer to Question (1).**

   Combining Theorem ($\star$) and Theorem ($\star\star$), we know the following:—

   Given q *column vectors* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *which are not necessarily linearly independent in the first place, if we know that*

   - r *of them, say,* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_r$*, are linearly independent over the reals, and*

- *the remaining column vectors, which are $\mathbf{u}_{r+1}, \mathbf{u}_{r+2}, \cdots, \mathbf{u}_q$ are linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_r$,*

then we may apply Theorem ($\star\star$) to conclude that

(a) $\text{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\}) = \text{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_r\})$,

and further apply Theorem ($\star$) to conclude that

(b) $\text{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\})$ *is an r-dimensional subspace of $\mathbb{R}^n$ over the reals, with a basis given by $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_r$.*

5. But such an answer to Question (1) will be 'practically useless' unless the 'practical' question below can be adequately answered:—

   Question (2).

   Given p column vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ which are not necessarily linearly independent in the first place:—

   (a) *How to 'extract' some of them which are themselves linearly independent over the reals and which span the others as linear combinations of them over the reals?*
   (b) *And how to do it methodically and efficiently, if it can be done at all?*

6. **Answer to Question (2).**

   An adequate answer to Question (2) is provided by the theoretical result below. Its proof is provided in the *appendix*.

   Theorem (1). ('Minimal spanning set' and dimension for a span of several column vectors.)

   Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q \in \mathbb{R}^n$, and $\mathcal{W} = \text{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\})$.

   Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_q\ ]$, and denote the rank of U by r.

   Then the statements below hold:—

   (a) $\dim(\mathcal{W}) = r$.
   (b) *From now on suppose $r \geq 1$.*
   Suppose $U^\sharp$ is a row-echelon form which is row-equivalent to U, and suppose the pivot columns of $U^\sharp$, from left to right, are the d_1-th, d_2-th, ..., d_r-th columns of $U^\sharp$.
   Then $\mathbf{u}_{d_1}, \mathbf{u}_{d_2}, \cdots, \mathbf{u}_{d_r}$ constitute a basis for $\mathcal{W}$ over the reals.
   (c) *Suppose U' is the reduced row-echelon form which is row-equivalent to U.*
   For each $j = 1, 2, \cdots, q$, denote the j-th column of U' by $\mathbf{u}'_j$, and denote the first r entries, from the top downwards, of $\mathbf{u}'_j$ by $\alpha_{1j}, \alpha_{2j}, \cdots, \alpha_{rj}$.
   Suppose the k-th column of U' is a free column in U'.
   Then $\mathbf{u}_k = \alpha_{1k}\mathbf{u}_{d_1} + \alpha_{2k}\mathbf{u}_{d_2} + \cdots + \alpha_{rk}\mathbf{u}_{d_r}$.

   Remark on terminology. In the context of Theorem (1), the set of column vectors

   $$\{\mathbf{u}_{d_1}, \mathbf{u}_{d_2}, \cdots, \mathbf{u}_{d_r}\},$$

   named in part (b) of the conclusion, is called a **minimal spanning set for $\mathcal{W}$ extracted from $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$**.

   - It is 'spanning' in the sense that using $\mathbf{u}_{d_1}, \mathbf{u}_{d_2}, \cdots, \mathbf{u}_{d_r}$ alone for forming linear combinations, we can still obtain every column vector which is spanned by $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.
   - It is 'minimal' in the sense that in order to be able to obtain all column vectors that can be spanned by $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$, we must not further 'exclude' any one amongst $\mathbf{u}_{d_1}, \mathbf{u}_{d_2}, \cdots, \mathbf{u}_{d_r}$.

7. **'Algorithm' associated with Theorem (1).**

   Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q \in \mathbb{R}^n$, and $\mathcal{W} = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\})$. We are going to determine the dimension of $\mathcal{W}$, and obtain a basis for $\mathcal{W}$ which is a minimal spanning set extracted from $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.

   Step (0). Inspect $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.

   If they are all zero vectors, declare $\dim(\mathcal{W}) = 0$ and stop.

   Otherwise go to Step (1).

   Step (1). Form the matrix $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_q\ ]$. Go to Step (2).

   Step (2). Obtain a row-echelon form $U^\sharp$ which is row-equivalent to U. Go to Step (3).

   Step (3). Inspect the matrix $U^\sharp$. Read off its rank r, and identify its pivot columns.

Conclude that $\dim(\mathcal{W}) = r$.

To obtain a basis for $\mathcal{W}$ which is a minimal spanning set extracted from $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$, go to step (4).

Step (4). Label the pivot columns of $U^\sharp$, from left to right, by $d_1, d_2, \cdots, d_r$.

Conclude that $\mathbf{u}_{d_1}, \mathbf{u}_{d_2}, \cdots, \mathbf{u}_{d_r}$ constitute a basis for $\mathcal{W}$ which is a minimal spanning set extracted from $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.

To know how each of the other column vectors amongst $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ is expressed as a linear combination of $\mathbf{u}_{d_1}, \mathbf{u}_{d_2}, \cdots, \mathbf{u}_{d_r}$, go to Step (5).

Step (5). Obtain the reduced row-echelon form U' which is row-equivalent to $U^\sharp$ and to U.

Suppose k is between 1 and q, and is not amongst $d_1, d_2, \cdots, d_r$.

Read off from the k-th column of U' the first r entries, from the top downwards, as $\alpha_{1k}, \alpha_{2k}, \cdots, \alpha_{rk}$.

Conclude that $\mathbf{u}_k = \alpha_{1k}\mathbf{u}_{d_1} + \alpha_{2k}\mathbf{u}_{d_2} + \cdots + \alpha_{rk}\mathbf{u}_{d_r}$.

8. **Example (1). (Illustrations on the algorithm associated to Theorem (1).)**

(a) *Let* $\mathbf{u}_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} -2 \\ 3 \\ -12 \end{bmatrix}$, *and* $\mathcal{W} = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\})$.

We want to find a basis for $\mathcal{W}$ *which is a minimal spanning set extracted from* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3\ ]$.

We obtain the reduced row-echelon form U' which is row-equivalent to U:

$$U = \begin{bmatrix} 0 & 1 & -2 \\ -1 & -2 & 3 \\ 2 & 7 & -12 \end{bmatrix} \longrightarrow \cdots\cdots\cdots \longrightarrow U' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

The rank of U' is 2, and the pivot columns are the 1-st and 2-nd columns.

Hence $\dim(\mathcal{W}) = 2$.

A basis for $\mathcal{W}$ is constituted by $\mathbf{u}_1, \mathbf{u}_2$.

As a bonus, we also obtain the vector equality $\mathbf{u}_3 = \mathbf{u}_1 - 2\mathbf{u}_2$.

(b) *Let* $\mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \mathbf{u}_5 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \mathbf{u}_6 = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$, *and* $\mathcal{W} = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6\})$.

We want to find a basis for $\mathcal{W}$ *which is a minimal spanning set extracted from* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6$.

Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4 \mid \mathbf{u}_5 \mid \mathbf{u}_6\ ]$.

We obtain the reduced row-echelon form U' which is row-equivalent to U:

$$U = \begin{bmatrix} 0 & 1 & 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 2 & 3 & 4 \\ -2 & -1 & -3 & 3 & 1 & 3 \end{bmatrix} \longrightarrow \cdots\cdots\cdots \longrightarrow U' = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 10 \\ 0 & 1 & 1 & 0 & 0 & -8 \\ 0 & 0 & 0 & 1 & 1 & 5 \end{bmatrix}.$$

The rank of U' is 3, and the pivot columns are the 1-st, 2-nd, and 4-th columns.

Hence $\dim(\mathcal{W}) = 3$.

A basis for $\mathcal{W}$ is constituted by $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_4$.

As a bonus, we also obtain the vector equalities $\mathbf{u}_3 = \mathbf{u}_1 + \mathbf{u}_2$, $\mathbf{u}_5 = \mathbf{u}_1 + \mathbf{u}_4$, and $\mathbf{u}_6 = 10\mathbf{u}_1 - 8\mathbf{u}_2 + 5\mathbf{u}_4$.

(c) *Let* $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -4 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -1 \\ -2 \\ 3 \\ 2 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 2 \\ 6 \\ -7 \\ 0 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} -7 \\ -18 \\ 23 \\ 7 \end{bmatrix}, \mathbf{u}_5 = \begin{bmatrix} -23 \\ -55 \\ 73 \\ 33 \end{bmatrix}$, *and* $\mathcal{W} = \mathrm{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\})$.

We want to find a basis for $\mathcal{W}$ *which is a minimal spanning set extracted from* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$.

Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4 \mid \mathbf{u}_5\ ]$.

We find the reduced row-echelon form U' which is row-equivalent to U:

$$U = \begin{bmatrix} 1 & -1 & 2 & -7 & -23 \\ 3 & -2 & 6 & -18 & -55 \\ -4 & 3 & -7 & 23 & 73 \\ 1 & 2 & 0 & 7 & 33 \end{bmatrix} \longrightarrow \cdots\cdots\cdots \longrightarrow U' = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}.$$

The rank of U' is 4, and the pivot columns are the 1-st, 2-nd, 3-rd, and 4-th columns.

Hence $\dim(\mathcal{W}) = 4$. (We further know that $\dim(\mathcal{W}) = \mathbb{R}^4$.)

A basis for $\mathcal{W}$ is constituted by $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$.

As a bonus, we also obtain the vector equality $\mathbf{u}_4 = \mathbf{u}_1 + 2\mathbf{u}_2 + 3\mathbf{u}_3 + 4\mathbf{u}_4$.

(d) *Let* $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 3 \\ 3 \\ 6 \\ -3 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ -3 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} -2 \\ -3 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{u}_5 = \begin{bmatrix} 1 \\ -3 \\ 10 \\ -5 \end{bmatrix}$, $\mathbf{u}_6 = \begin{bmatrix} -3 \\ -4 \\ -3 \\ -1 \end{bmatrix}$, *and*

$\mathcal{W} = \text{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6\})$.

We want to find a basis for $\mathcal{W}$ *which is a minimal spanning set extracted from* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6$.

Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4 \mid \mathbf{u}_5 \mid \mathbf{u}_6\ ]$.

We find the reduced row-echelon form U' which is row-equivalent to U:

$$U = \begin{bmatrix} 1 & 3 & 1 & -2 & 1 & -3 \\ 1 & 3 & 2 & -3 & -3 & -4 \\ 2 & 6 & 1 & -2 & 10 & -3 \\ -1 & -3 & -3 & 1 & -5 & -1 \end{bmatrix} \longrightarrow \cdots\cdots\cdots \longrightarrow U' = \begin{bmatrix} 1 & 3 & 0 & 0 & 9 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The rank of U' is 3, and the pivot columns are the 1-st, 3-rd, and 4-th columns.

Hence $\dim(\mathcal{W}) = 3$.

A basis for $\mathcal{W}$ is constituted by $\mathbf{u}_1, \mathbf{u}_3, \mathbf{u}_4$.

As a bonus, we also obtain the vector equalities $\mathbf{u}_2 = 3\mathbf{u}_1$, $\mathbf{u}_5 = 9\mathbf{u}_1 + 4\mathbf{u}_4$, and $\mathbf{u}_6 = \mathbf{u}_3 + 2\mathbf{u}_4$.

9. Theorem (1) will be applied in the proof of some important theoretical results about the rank of matrices. One of them will be introduced shortly.

To prepare the way for formulating these results, we need to introduce some new concepts first. They are built upon the notion of span.

Definition. (Column space and row space for matrices.)

Let A *be a* $(p \times q)$*-matrix with real entries.*

(1) *Denote the columns of* A*, from left to right, by* $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_q$.

The **column space** *of* A *is defined to be* $\mathsf{Span}\ (\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_q\})$. *It is denoted by* $\mathcal{C}(A)$.

(2) *Write* $B = A^t$*, and denote the columns of* B*, from left to right, by* $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_p$.

(So $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_p$ *are the transposes of the respective rows of* A *from top to bottom.)*

The **row space** *of* A *is defined to be* $\mathsf{Span}\ (\{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_p\})$. *It is denoted by* $\mathcal{R}(A)$.

Remark. In plain words:—

(a) The column space of A is the subspace of $\mathbb{R}^q$ consisting of those and only those column vectors which can be obtained as linear combinations of the columns of A.

(b) The row space of A is the subspace of $\mathbb{R}^p$ consisting of those and only those column vectors which can be obtained as linear combinations of the transposes of the rows of A.

Because of the behaviour of the 'transpose operation', the row space of A consists of those and only those transposes of row vectors which can be obtained as linear combinations of the rows of A.

10. **Example (2). (Illustration on column space and row space for non-square matrices.)**

Let $A = \begin{bmatrix} 1 & 3 & 1 & -2 & 1 & -3 \\ 1 & 3 & 2 & -3 & -3 & -4 \\ 2 & 6 & 1 & -2 & 10 & -3 \\ -1 & -3 & -3 & 1 & -5 & -1 \end{bmatrix}$.

(a) The columns of A, from left to right, are given by

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 3 \\ 3 \\ 6 \\ -3 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ -3 \end{bmatrix}, \mathbf{a}_4 = \begin{bmatrix} -2 \\ -3 \\ -2 \\ 1 \end{bmatrix}, \mathbf{a}_5 = \begin{bmatrix} 1 \\ -3 \\ 10 \\ -5 \end{bmatrix}, \mathbf{a}_6 = \begin{bmatrix} -3 \\ -4 \\ -3 \\ -1 \end{bmatrix}.$$

(So $A = [\ \mathbf{a}_1 \mid \mathbf{a}_2 \mid \mathbf{a}_3 \mid \mathbf{a}_4 \mid \mathbf{a}_5 \mid \mathbf{a}_6\ ]$.)

Then $\mathcal{C}(A) = \mathsf{Span}\ (\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6\})$.

(Note that $\mathcal{C}(A)$ is a subspace of $\mathbb{R}^4$ over the reals.)

(b) Write $B = A^t$. The columns of B, from left to right, are given by

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ -2 \\ 1 \\ -3 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ -2 \\ 1 \\ -3 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 1 \\ 3 \\ 2 \\ -3 \\ -3 \\ -4 \end{bmatrix}, \mathbf{b}_4 = \begin{bmatrix} -1 \\ -3 \\ -3 \\ 1 \\ -5 \\ -1 \end{bmatrix}.$$

(So $A^t = B = [\ \mathbf{b}_1 \mid \mathbf{b}_2 \mid \mathbf{b}_3 \mid \mathbf{b}_4\ ]$, $A = B^t = \begin{bmatrix} \mathbf{b}_1^t \\ \hline \mathbf{b}_2^t \\ \hline \mathbf{b}_3^t \\ \hline \mathbf{b}_4^t \end{bmatrix}$.)

Then $\mathcal{R}(A) = \mathsf{Span}\ (\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\})$.

(Note that $\mathcal{R}(A)$ is a subspace of $\mathbb{R}^6$ over the reals.)

11. **Distinction between column space and row space for an arbitrary matrix.**

Example (2) reminds us that the column space and the row space of a non-square matrix are necessarily distinct from each other as sets. They are not even 'comparable' because they are subspaces of distinct $\mathbb{R}^n$'s.

Example (3) tells us that even for a square matrix, its column space and its row space can be different sets, and should be distinguished from each other.

12. **Example (3). (Illustration on column space and row space for square matrices.)**

Let $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Note that last column of A is a linear combination of the first three columns of A.

Then we have

$$\mathcal{C}(A) = \text{Span}(\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\2\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\3\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\0\end{bmatrix}\}) = \text{Span}(\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\2\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\3\\0\end{bmatrix}\})$$

.

Note that every column vector belonging to $\mathcal{C}(A)$ will have 0 as its last entry.

(b) Note that the last row of A is a linear combination of the first three rows of A.

Then we have

$$\mathcal{R}(A) = \text{Span}(\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\2\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\3\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\end{bmatrix}\}) = \text{Span}(\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\2\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\3\\1\end{bmatrix}\}).$$

Note that $\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$ belongs to $\mathcal{R}(A)$, and does not belong to $\mathcal{C}(A)$.

Hence $\mathcal{R}(A)$ and $\mathcal{C}(A)$ are not equal to each other as sets (and have no chance of being the same subspace of $\mathbb{R}^4$ over the reals).

13. We now state a few theoretical results concerned with column space and row space for matrices, not strictly according to the 'logical order' (in which one is used in the proof of the other).

14. The most important result that we can state that we can state with the terminologies we have introduced up to this point is Theorem (2). It links up two key numbers associated with the same arbitrarily given matrix, namely the respective dimensions of its column space and its row space, through rank of the given matrix:—

Theorem (2). (Equality amongst rank, 'column rank' and 'row rank' for an arbitrarily given matrix.)

Let A be an $(n \times p)$-matrix with real entries.

Suppose A is of rank r.

Then $\dim(\mathcal{C}(A)) = r = \dim(\mathcal{R}(A))$.

Comment on the content of Theorem (2).

Recall that $\mathcal{C}(A)$, $\mathcal{R}(A)$ are distinct objects, and in general incomparable with each other. There is no reason to expect their dimensions to be the same as each other. This is why Theorem (2) is highly non-trivial.

Remark on terminologies.

- The number $\dim(\mathcal{C}(A))$ is very often referred to as the **column rank of** A.
- Then number $\dim(\mathcal{R}(A))$ is very often referred to as the **row rank of** A.

15. The proof of Theorem (2) is provided in the *appendix*. Its relies on Theorem (1), and the other theoretical results about the notion of row space and column space that we are going to state below.

Example (4) may serve as 'evidence' in support of Theorem (2), (provided we trust the validity of Theorem (1)).

Example (4). ('Evidence' in support of Theorem (2).)

(a) *Let* $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

We apply Theorem (1) to evaluate the respective dimensions of the column space and the row space of A.

A is row-equivalent to the reduced row-echelon form A':—

$$A \longrightarrow \cdots \longrightarrow A' = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence the rank of A is 3.

There are three pivot columns in A', namely, the 1-st, 2-nd, 3-rd columns of A'.

i. By definition, $\mathcal{C}(A)$ is the span of the columns of A. Then, by Theorem (1), a basis for $\mathcal{C}(A)$ is given by the 1-st, 2-nd, 3-rd columns of A.
Hence $\dim(\mathcal{C}(A)) = 3$.

ii. We have the sequence of row operations joining $B = A^t$ with the reduced row-echelon form B' which is row-equivalent to B:

$$B = A^t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \longrightarrow \cdots \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of B' is 3.
There are three pivot columns in B', namely, the 1-st, 2-nd, 3-rd columns of B'.
By definition, $\mathcal{R}(A)$ is the span of the columns of A^t. Then, by Theorem (1), a basis for $\mathcal{R}(A)$ is given by the 1-st, 2-nd, 3-rd columns of A^t.
Hence $\dim(\mathcal{R}(A)) = 3$.

(b) *Let* $A = \begin{bmatrix} 1 & 3 & 1 & -2 & 1 & -3 \\ 1 & 3 & 2 & -3 & -3 & -4 \\ 2 & 6 & 1 & -2 & 10 & -3 \\ -1 & -3 & -3 & 1 & -5 & -1 \end{bmatrix}$.

We apply Theorem (1) to evaluate the respective dimensions of the column space and the row space of A.

A is row-equivalent to the reduced row-echelon form A':—

$$A \longrightarrow \cdots \longrightarrow A' = \begin{bmatrix} 1 & 3 & 0 & 0 & 9 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence the rank of A is 3.

There are three pivot columns in A', namely, the 1-st, 3-rd, 4-th columns of A'.

i. By definition, $\mathcal{C}(A)$ is the span of the columns of A. Then, by Theorem (1), a basis for $\mathcal{C}(A)$ is given by the 1-st, 3-rd, 4-th columns of A.
Hence $\dim(\mathcal{C}(A)) = 3$.

ii. We have the sequence of row operations joining $B = A^t$ with the reduced row-echelon form B' which is row-equivalent to B:

$$B = A^t = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 3 & 3 & 6 & -3 \\ 1 & 2 & 1 & -3 \\ -2 & -3 & -2 & 1 \\ 1 & -3 & 10 & -5 \\ -3 & -4 & -3 & -1 \end{bmatrix} \longrightarrow \cdots \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of B' is 3.
There are three pivot columns in B', namely, the 1-st, 2-nd, 3-rd columns of B'.
By definition, $\mathcal{R}(A)$ is the span of the columns of A^t. Then, by Theorem (1), a basis for $\mathcal{R}(A)$ is given by the 1-st, 2-nd, 3-rd columns of A^t.
Hence $\dim(\mathcal{R}(A)) = 3$.

16. The result below, Lemma (3), follows immediately from the definitions for the notions of set equality, column space and row space. Its proof is almost an easy word-game, and is therefore left as an exercise.

Lemma (3).

Suppose A is a $(p \times q)$-matrix with real entries. Then the statements below hold:—

(a) $\mathcal{C}(A^t) = \mathcal{R}(A)$, *and* $\mathcal{C}(A) = \mathcal{R}(A^t)$.

(b) i. $\mathcal{C}(A) = \left\{ \mathbf{y} \in \mathbb{R}^p \;\middle|\; \text{There exist some } \mathbf{t} \in \mathbb{R}^q \text{ such that } \mathbf{y} = A\mathbf{t}. \right\}$.

ii. $\mathcal{R}(A) = \left\{ \mathbf{x} \in \mathbb{R}^q \;\middle|\; \text{There exist some } \mathbf{s} \in \mathbb{R}^p \text{ such that } \mathbf{x}^t = \mathbf{s}^t A. \right\}$.

Remark. In the argument for part (b), we need to use the 'dictionary' between linear combinations and matrix-vector product.

17. Theorem (4) is concerned with the interaction between the notion of column space for matrices and that of left-multiplication by invertible matrices. Its proof is provided in the *appendix*.

Theorem (4).

Let A, B be matrices with n rows and with real entries.

Suppose H is an invertible $(n \times n)$-square matrix with real entries. Then the statements below are logically equivalent:—

(1) $\mathcal{C}(A) = \mathcal{C}(B)$.

(2) $\mathcal{C}(HA) = \mathcal{C}(HB)$.

18. **Comment on the content of Theorem (4).**

We can re-interpret the content of Theorem (4) in terms of row-equivalence and row operations, through the 'dictionary' between application of sequences of row operations and left-multiplication by invertible square matrices:—

Suppose A, B are respectively row-equivalent to $\widetilde{A}, \widetilde{B}$ under the same sequence of row operations, and also suppose $\mathcal{C}(A) = \mathcal{C}(B)$.

Then as a consequence, $\mathcal{C}\left(\widetilde{A}\right) = \mathcal{C}\left(\widetilde{B}\right)$.

19. Theorem (5) is concerned with the interaction between the notion of column space for matrices and that of right-multiplication by invertible matrices.

Theorem (5).

Let B be a matrix with n columns and with real entries, and G be an $(n \times n)$-square matrix with real entries.

Suppose G in invertible. Then $\mathcal{C}(B) = \mathcal{C}(BG)$.

Remark. The proof of Theorem (5) is provided in the *appendix*. Like the proof of Theorem (5), this is also an exercise on the definition for the notion of set equality and that of invertibility for square matrices.

20. We can re-formulate Theorem (5) in terms of row spaces.

Theorem (6). (Corollary to Theorem (5).)

Let A be a matrix with n rows and with real entries, and H be an $(n \times n)$-square matrix with real entries.

Suppose H in invertible. Then $\mathcal{R}(A) = \mathcal{R}(HA)$.

Proof of Theorem (6).

Let A be a matrix with n rows and with real entries, and H be an $(n \times n)$-square matrix with real entries.

Suppose H in invertible.

Note that A^t is a matrix with n columns and with real entries, and H^t is an $(n \times n)$-square matrix with real entries.

By Lemma (3) and Theorem (5), we have $\mathcal{R}(A) = \mathcal{C}(A^t) = \mathcal{C}(A^t H^t) = \mathcal{C}((HA)^t) = \mathcal{R}(HA)$.

21. **Comment on the content of Theorem (6).**

We can re-interpret the content of Theorem (6) in terms of row-equivalence and row operations, through the 'dictionary' between application of sequences of row operations and left-multiplication by invertible square matrices:—

Suppose A is row-equivalent to $\widetilde{A}$. Then as a consequence, $\mathcal{R}(A) = \mathcal{R}\left(\widetilde{A}\right)$.

Or simply:—

The row space of an arbitrary matrix remains unchanged irrespective of what row operations are applied on the matrix concerned.

22. We formulate the above observation as the result below. (More formally writing up the above observation, we have the formal proof of the result.)

Theorem (7). (Corollary to Theorem (6).)

Let $A, \widetilde{A}$ be matrices with real entries of the same size.

Suppose A is row-equivalent to $\widetilde{A}$.

Then $\mathcal{R}(A) = \mathcal{R}\left(\widetilde{A}\right)$.

23. Theorem (1) and Theorem (7) are the main machinery that will be used in the proof of Theorem (2), which is provided in the *appendix.*

The example below illustrates the content of Theorem (2) and the way Theorem (1) and Theorem (7) are used in the proof of Theorem (2).

Example (5). (Illustration of the content of Theorem (2) and the idea in the proof of Theorem (2).)

Let $A = \begin{bmatrix} 1 & 3 & 1 & -2 & 1 & -3 \\ 1 & 3 & 2 & -3 & -3 & -4 \\ 2 & 6 & 1 & -2 & 10 & -3 \\ -1 & -3 & -3 & 1 & -5 & -1 \end{bmatrix}$.

Note that A is row-equivalent to the reduced row-echelon form A' under some sequence of row operations:—

$$A = \begin{bmatrix} 1 & 3 & 1 & -2 & 1 & -3 \\ 1 & 3 & 2 & -3 & -3 & -4 \\ 2 & 6 & 1 & -2 & 10 & -3 \\ -1 & -3 & -3 & 1 & -5 & -1 \end{bmatrix} \longrightarrow \cdots\cdots\cdots \longrightarrow A' = \begin{bmatrix} 1 & 3 & 0 & 0 & 9 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that the rank of A is 3. There are exactly:—

- 3 non-zero rows in A', namely, the 1-st, 2-nd, 3-rd rows of A'.
- 3 leading ones in A', namely, the $(1,1)$-th entry, the $(2,3)$-th entry, the $(3,4)$-th entry of A'.
- 3 pivot columns in A', namely, the 1-st, 3-rd, 4-th columns of A'.

(a) Denote the columns of A, from left to right, by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6$.
By Lemma (3), we have $\mathcal{C}(A) = \text{Span}(\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6\})$.
According to Theorem (1), $\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4$ constitute a basis for $\text{Span}(\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6\})$ over the reals.
Then $\dim(\mathcal{C}(A)) = \dim(\text{Span}(\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6\})) = 3$.

(b) By Theorem (7), we have $\mathcal{R}(A) = \mathcal{R}(A')$.
Write $B = (A')^t$. The only non-zero columns of B are its first three columns, given by

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 9 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 4 \\ 2 \end{bmatrix}.$$

We verify that $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ constitute a basis for $\mathcal{R}(A')$ over the reals:—

i. Note that:—
   - the first non-zero entry of $\mathbf{b}_2$ is strictly below the first non-zero entry of $\mathbf{b}_1$, and
   - the first non-zero entry of $\mathbf{b}_3$ is strictly below the first non-zero entry of $\mathbf{b}_2$.

   Then (with this) we deduce that $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are linearly independent over the reals.

ii. Now we have, according to Lemma (3) and Theorem ($\star\star$),

$$\mathcal{R}(A') = \mathcal{C}(B) = \text{Span}(\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{0}_6\}) = \text{Span}(\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}).$$

   It follows that $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ constitute a basis for $\mathcal{R}(A')$ over the reals.

Hence $\dim(\mathcal{R}(A)) = \dim(\mathcal{R}(A')) = 3$.