4.5.1 Appendix: Proofs of basic theoretical results concerned with bases and dimensions for general subspaces of column vectors.

0. The material in this appendix is supplementary.

1. We have introduced these results:—

   Theorem (1). (Upper bound of number of column vectors in a basis for a subspace of $\mathbb{R}^n$ over the reals.)

   Every basis for any subspace of $\mathbb{R}^n$ over the reals has at most n column vectors.

   Theorem (2). (Existence of basis for an arbitrary non-zero subspace of $\mathbb{R}^n$ over the reals.)

   Suppose $\mathcal{V}$ is a non-zero subspace of $\mathbb{R}^n$ over the reals. Then there is a basis for $\mathcal{V}$ over the reals which consists of at least one and at most n column vectors with n real entries.

2. We are going to give a proof for Theorem (2). In its argument, two results on linear dependence and linear independence that we have learnt earlier will play a crucial role. They are Lemma ($*$) and Lemma ($**$).

   Lemma ($*$).

   Let $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_k, \mathbf{v} \in \mathbb{R}^n$.

   Suppose $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_k$ are linearly independent over the reals.

   Then the statements below are logically equivalent:—

   ($*_1$) $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_k, \mathbf{v}$ *are linearly independent over the reals.*

   ($*_2$) $\mathbf{v}$ *is not a linear combination of* $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_k$ *over the reals.*

   Lemma ($**$).

   Let $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_\ell \in \mathbb{R}^n$. Suppose $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_\ell$ are linearly independent over the reals. Then $\ell \leq n$.

   Remark. Recall that Lemma ($**$) is essentially what Theorem (1) is about.

3. **Proof of Theorem (2).**

   Suppose $\mathcal{V}$ is a non-zero subspace of $\mathbb{R}^n$ over the reals.

   (a) By assumption we may pick some $\mathbf{u}_1 \in \mathcal{V}$ so that $\sqcap_1 \neq \mathbf{0}_n$.
   $\mathbf{u}_1$ is linearly independent over the reals.
   If every column vector belonging to $\mathcal{V}$ is a linear combination of $\mathbf{u}_1$ over the reals, then $\mathbf{u}_1$ constitutes a basis for $\mathcal{V}$ over the reals.

   (b) Suppose that not every column vector belonging to $\mathcal{V}$ is a linear combination of $\mathbf{u}_1$ over the reals.
   Then we may pick some $\mathbf{u}_2 \in \mathcal{V}$ so that $\mathbf{u}_2$ is not a linear combination of $\mathbf{u}_1$ over the reals.
   By Lemma ($*$), $\mathbf{u}_1, \mathbf{u}_2$ are linearly independent over the reals.
   If every column vector belonging to $\mathcal{V}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2$ over the reals, then $\mathbf{u}_1, \mathbf{u}_2$ constitute a basis for $\mathcal{V}$ over the reals.

   (c) Suppose that not every column vector belonging to $\mathcal{V}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2$ over the reals.
   Then we may pick some $\mathbf{u}_3 \in \mathcal{V}$ so that $\mathbf{u}_3$ is not a linear combination of $\mathbf{u}_1, \mathbf{u}_2$ over the reals.
   By Lemma ($*$), $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly independent over the reals.
   If every column vector belonging to $\mathcal{V}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ over the reals, then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ constitute a basis for $\mathcal{V}$ over the reals.

   (d) Suppose j is any one integer greater than j.
   Under the assumption that not every column vector belonging to $\mathcal{V}$ is a linear combination of $\mathbf{u}_1, \cdots, \mathbf{u}_{j-1}$ over the reals, we may pick some $\mathbf{u}_j \in \mathcal{V}$ so that $\mathbf{u}_j$ is not a linear combination of $\mathbf{u}_1, \cdots, \mathbf{u}_{j-1}$ over the reals.
   By Lemma ($*$), these same column vectors $\mathbf{u}_1, \cdots \mathbf{u}_{j-1}, \mathbf{u}_j$ are linearly independent over the reals.

   (e) We have obtained, in succession, some sequence of column vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots \in \mathcal{V}$, which are linearly independent vectors in $\mathbb{R}^n$.
   By Lemma ($**$), this sequence terminates at $\mathbf{u}_p$, for some integer $p \leq n$.
   It is then necessarily true that every column vector belonging to $\mathcal{V}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ over the reals; (otherwise, we could repeat the construction to obtain $\mathbf{u}_{p+1}$).
   It follows that the p vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ constitute a basis for $\mathcal{V}$ over the reals.

4. We now proceed to prove the Replacement Theorem:—

Theorem (7). (Replacement Theorem.)

Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$ over the reals. Let p, q be positive integers.

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p, \mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q \in \mathcal{W}$.

Suppose $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$ constitute a basis for $\mathcal{W}$ over the reals.

Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are linearly independent over the reals.

Then:—

(a) *the inequality $p \leq q$ holds, and*

(b) *$\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$, and some $q-p$ column vectors amongst $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$ together, constitute a basis of $\mathcal{W}$ over the reals.*

5. To prepare the way for an argument for Theorem (7), we prove two weaker results first.

Lemma (7'). (Baby version of Replacement Theorem.)

Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$ over the reals. Suppose $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ constitute a basis for $\mathcal{W}$ over the reals.

Let $\mathbf{u} \in \mathbb{R}^n$. Suppose $\mathbf{u} \neq \mathbf{0}_n$, and $\mathbf{u}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ over the reals.

Then, the column vector $\mathbf{u}$ and some $k-1$ column vectors amongst $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ together, constitutes a basis of $\mathcal{W}$ over the reals.

6. **Proof of Lemma (7').**

Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$. Suppose $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ constitute a basis for $\mathcal{W}$ over the reals.

Let $\mathbf{u} \in \mathbb{R}^n$. Suppose $\mathbf{u} \neq \mathbf{0}_n$, and $\mathbf{u}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ over the reals.

By assumption, there exist some $\alpha_1, \alpha_2, \cdots, \alpha_k \in \mathbb{R}$ such that $\mathbf{u} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k$.

By assumption $\mathbf{u} \neq \mathbf{0}_n$. Then at least one of $\alpha_1, \alpha_2, \cdots, \alpha_k$ is non-zero.

Without loss of generality, suppose $\alpha_1 \neq 0$.

(Otherwise, choose the first ℓ for which α_ℓ is non-zero. Then re-label $\alpha_1, \mathbf{v}_1$ as $\alpha_\ell, \mathbf{v}_\ell$ respectively, and $\alpha_\ell, \mathbf{v}_\ell$ as $\alpha_1, \mathbf{v}_1$ respectively.)

We verify that $\mathbf{u}, \mathbf{v}_2, \mathbf{v}_3, \cdots, \mathbf{v}_k$ constitute a basis for $\mathcal{W}$ over the reals:

- [We verify (BL): '*$\mathbf{u}, \mathbf{v}_2, \mathbf{v}_3, \cdots, \mathbf{v}_k$ are linearly independent over the reals.*']
  Pick any $\beta, \gamma_2, \gamma_3, \cdots, \gamma_k \in \mathbb{R}$.
  Suppose $\beta\mathbf{u} + \gamma_2\mathbf{v}_2 + \gamma_3\mathbf{v}_3 + \cdots + \gamma_k\mathbf{v}_k = \mathbf{0}_n$.
  Then
  $$\begin{aligned}\mathbf{0}_n &= \beta(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k) + \gamma_2\mathbf{v}_2 + \gamma_3\mathbf{v}_3 + \cdots + \gamma_k\mathbf{v}_k \\ &= \beta\alpha_1\mathbf{v}_1 + (\beta\alpha_2 + \gamma_2)\mathbf{v}_2 + (\beta\alpha_3 + \gamma_3)\mathbf{v}_3 + \cdots + (\beta\alpha_k + \gamma_k)\mathbf{v}_k\end{aligned}$$
  By assumption $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ are linearly independent over the reals.
  Then
  $$\beta\alpha_1 = 0 = \beta\alpha_2 + \gamma_2 = \beta\alpha_3 + \gamma_3 = \cdots = \beta\alpha_k + \gamma_k.$$
  Recall that $\alpha_1 \neq 0$. Then since $\beta\alpha_1 = 0$, we have $\beta = 0$.
  Therefore $\gamma_2 = \gamma_3 = \cdots = \gamma_k = 0$ also.
  It follows that $\mathbf{u}, \mathbf{v}_2, \mathbf{v}_3, \cdots, \mathbf{v}_k$ are linearly independent over the reals.
- [We verify (BS): '*Every column vector belonging to $\mathcal{W}$ is a linear combination of $\mathbf{u}, \mathbf{v}_2, \mathbf{v}_3, \cdots, \mathbf{v}_k$ over the reals.*']
  Pick any $\mathbf{x} \in \mathbb{R}^n$. Suppose $\mathbf{x} \in \mathcal{W}$.
  By assumption, $\mathbf{x}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots, \mathbf{v}_k$ over the reals.
  Then there exist some $\delta_1, \delta_2, \cdots, \delta_k \in \mathbb{R}$ such that $\mathbf{x} = \delta_1\mathbf{v}_1 + \delta_2\mathbf{v}_2 + \cdots + \delta_k\mathbf{v}_k$.
  Now recall that $\mathbf{u} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k$.
  Then $\mathbf{v}_1 = \dfrac{1}{\alpha_1}\mathbf{u} - \dfrac{\alpha_2}{\alpha_1}\mathbf{v}_2 - \dfrac{\alpha_3}{\alpha_1}\mathbf{v}_3 - \cdots - \dfrac{\alpha_k}{\alpha_1}\mathbf{v}_k$.
  Therefore, for the same $\mathbf{x}$, we have
  $$\begin{aligned}\mathbf{x} &= \delta_1\left(\frac{1}{\alpha_1}\mathbf{u} - \frac{\alpha_2}{\alpha_1}\mathbf{v}_2 - \frac{\alpha_3}{\alpha_1}\mathbf{v}_3 - \cdots - \frac{\alpha_k}{\alpha_1}\mathbf{v}_k\right) + \delta_2\mathbf{v}_2 + \cdots + \delta_k\mathbf{v}_k \\ &= \frac{\delta_1}{\alpha_1}\mathbf{u} + \left(\delta_2 - \frac{\delta_1\alpha_2}{\alpha_1}\right)\mathbf{v}_2 + \left(\delta_3 - \frac{\delta_1\alpha_3}{\alpha_1}\right)\mathbf{v}_3 + \cdots + \left(\delta_k - \frac{\delta_1\alpha_k}{\alpha_1}\right)\mathbf{v}_k\end{aligned}$$

It follows that $\mathbf{u}, \mathbf{v}_2, \mathbf{v}_3, \cdots, \mathbf{v}_k$ constitute a basis for $\mathcal{W}$ over the reals.

7. **Theorem (7"). (Weaker version of Replacement Theorem.)**

   Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$ over the reals. Let p be a positive integer, and s be a non-negative integer.

   Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p, \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p, \mathbf{v}_{p+1}, \cdots, \mathbf{v}_{p+s} \in \mathcal{W}$.

   Suppose $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p, \mathbf{v}_{p+1}, \cdots, \mathbf{v}_{p+s}$ constitute a basis for $\mathcal{W}$ over the reals.

   Further suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are linearly independent over the reals.

   Then, $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$, and some s column vectors amongst $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p, \mathbf{v}_{p+1}, \cdots, \mathbf{v}_{p+s}$ together, constitute a basis of $\mathcal{W}$ over the reals.

8. **Proof of Theorem (7").**

   Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$ over the reals. Suppose $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p, \mathbf{v}_{p+1}, \cdots, \mathbf{v}_{p+s}$ constitute a basis for $\mathcal{W}$.

   Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p \in \mathcal{W}$. Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are linearly independent over the reals.

   (a) $\mathbf{u}_1$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{p+s}$ over reals. Moreover, $\mathbf{u}_1 \neq \mathbf{0}_n$. (Why?)
   We apply Lemma (7'):—
   After relabelling the indices of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{p+s}$ if necessary, we obtain some base for $\mathcal{W}$ over the reals, given by $\mathbf{u}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots, \mathbf{v}_p, \mathbf{v}_{p+1} \cdots, \mathbf{v}_{p+s}$.

   (b) Suppose $1 \leq j < p$, and suppose that after relabelling the indices of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{p+s}$ if necessary, we have obtained some basis for $\mathcal{W}$ over the reals, given by $\mathbf{u}_1, \cdots, \mathbf{u}_j, \mathbf{v}_{j+1}, \cdots, \mathbf{v}_{p+s}$.

   i. By the definition of the notion of basis, $\mathbf{u}_{j+1}$ is a linear combination of $\mathbf{u}_1, \cdots, \mathbf{u}_j, \mathbf{v}_{j+1}, \mathbf{v}_{j+2}, \cdots, \mathbf{v}_{p+s}$ over the reals.
   Then there exist some $\kappa_1, \cdots, \kappa_j, \lambda, \mu_{j+2}, \cdots, \cdots, \mu_{p+s} \in \mathbb{R}$ such that

   $$\mathbf{u}_{j+1} = \kappa_1 \mathbf{u}_1 + \cdots + \kappa_j \mathbf{u}_j + \lambda \mathbf{v}_{j+1} + \mu_{j+2} \mathbf{v}_{j+2} + \cdots + \mu_{p+s} \mathbf{v}_{p+s}.$$

   ii. Note that $\lambda, \mu_{j+2}, \cdots, \cdots, \mu_{p+s}$ are not all zero. Justification:—

   * Suppose $\lambda, \mu_{j+2}, \cdots, \cdots, \mu_{p+s}$ were all zero.
   Then the equality $\mathbf{u}_{j+1} = \kappa_1 \mathbf{u}_1 + \kappa_2 \mathbf{u}_2 + \cdots + \kappa_j \mathbf{u}_j$ would hold.
   Such an equality is impossible because $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are assumed to be linearly independent over the reals.

   iii. Without loss of generality, suppose $\lambda \neq 0$.
   We verify that $\mathbf{u}_1, \cdots, \mathbf{u}_j, \mathbf{u}_{j+1}, \mathbf{v}_{j+2} \cdots, \mathbf{v}_{p+s}$ constitute a basis for V:

   - [We verify (BL): '*$\mathbf{u}_1, \cdots, \mathbf{u}_j, \mathbf{u}_{j+1}, \mathbf{v}_{j+2}, \cdots, \mathbf{v}_{p+s}$ are linearly independent over the reals.*']
     Pick any $\alpha_1, \cdots, \alpha_j, \beta, \gamma_{j+2}, \cdots, \gamma_{p+s} \in \mathbb{R}$.
     Suppose $\alpha_1 \mathbf{u}_1 + \cdots + \alpha_j \mathbf{u}_j + \beta \mathbf{u}_{j+1} + \gamma_{j+2} \mathbf{v}_{j+2} + \cdots + \gamma_{p+s} \mathbf{v}_{p+s} = \mathbf{0}_n$.
     Then

     $$\begin{aligned} \mathbf{0}_n &= \alpha_1 \mathbf{u}_1 + \cdots + \alpha_j \mathbf{u}_j \\ &\qquad + \beta(\kappa_1 \mathbf{u}_1 + \cdots + \kappa_j \mathbf{u}_j + \lambda \mathbf{v}_{j+1} + \mu_{j+2} \mathbf{v}_{j+2} + \cdots + \mu_{p+s} \mathbf{v}_{p+s}) \\ &\qquad\quad + \gamma_{j+2} \mathbf{v}_{j+2} + \cdots + \gamma_{p+s} \mathbf{v}_{p+s} \\ &= (\beta\kappa_1 + \alpha_1)\mathbf{u}_1 + \cdots + (\beta\kappa_j + \alpha_j)\mathbf{u}_j + \beta\lambda \mathbf{v}_{j+1} + (\beta\mu_{j+2} + \gamma_{j+2})\mathbf{v}_{j+2} + \cdots + (\beta\mu_{p+s} + \gamma_{p+s})\mathbf{v}_{p+s} \end{aligned}$$

     Note that $\mathbf{u}_1, \cdots, \mathbf{u}_j, \mathbf{v}_{j+1}, \cdots, \mathbf{v}_{p+s}$ are linearly independent. Then

     $$\beta\lambda = 0 = \beta\kappa_1 + \alpha_1 = \cdots = \beta\kappa_j + \alpha_j = \beta\mu_{j+2} + \gamma_{j+2} = \cdots = \beta\mu_{p+s} + \gamma_{p+s}.$$

     Recall that $\lambda \neq 0$. Then since $\beta\lambda = 0$, we have $\beta = 0$.
     Therefore $\alpha_1 = \cdots = \alpha_j = \gamma_{j+2} = \cdots = \gamma_{p+s} = 0$ also.
     It follows that $\mathbf{u}_1, \cdots, \mathbf{u}_j, \mathbf{u}_{j+1}, \mathbf{v}_{j+2}, \cdots, \mathbf{v}_{p+s}$ are linearly independent over the reals.
   - [We verify (BS): '*Every column vector belonging to $\mathcal{W}$ is a linear combination of $\mathbf{u}_1, \cdots, \mathbf{u}_j, \mathbf{u}_{j+1}, \mathbf{v}_{j+2}, \cdots, \mathbf{v}_{p+s}$ over the reals.*']
     Pick any $\mathbf{x} \in \mathbb{R}^n$. Suppose $\mathbf{x} \in \mathcal{W}$.
     By assumption, $\mathbf{x}$ is a linear combination of $\mathbf{u}_1, \cdots, \mathbf{u}_j, \mathbf{v}_{j+1}, \mathbf{v}_{j+2} \cdots, \mathbf{v}_{p+s}$ over the reals.
     Then there exist some $\delta_1, \cdots, \delta_j, \delta_{j+1}, \delta_{j+2}, \cdots, \delta_{p+s} \in \mathbb{R}$ such that

     $$\mathbf{x} = \delta_1 \mathbf{u}_1 + \cdots + \delta_j \mathbf{u}_j + \delta_{j+1} \mathbf{v}_{j+1} + \delta_{j+2} \mathbf{v}_{j+2} + \cdots + \delta_{p+s} \mathbf{v}_{p+s}.$$

Now recall that

$$\mathbf{u}_{j+1} = \kappa_1\mathbf{u}_1 + \cdots + \kappa_j\mathbf{u}_j + \lambda\mathbf{v}_{j+1} + \mu_{j+2}\mathbf{v}_{j+2} + \cdots + \mu_{p+s}\mathbf{v}_{p+s}.$$

Then $\mathbf{v}_{j+1} = \dfrac{1}{\lambda}\mathbf{u}_{j+1} - \dfrac{\kappa_1}{\lambda}\mathbf{u}_1 - \cdots - \dfrac{\kappa_j}{\lambda}\mathbf{u}_j - \dfrac{\mu_{j+2}}{\lambda}\mathbf{v}_{j+2} - \cdots - \dfrac{\mu_{p+s}}{\lambda}\mathbf{v}_{p+s}$.
Therefore, for the same $\mathbf{x}$, we have

$$\begin{aligned}\mathbf{x} &= \delta_1\mathbf{u}_1 + \cdots + \delta_j\mathbf{u}_j \\ &\quad +\delta_{j+1}\left(\frac{1}{\lambda}\mathbf{u}_{j+1} - \frac{\kappa_1}{\lambda}\mathbf{u}_1 - \cdots - \frac{\kappa_j}{\lambda}\mathbf{u}_j - \frac{\mu_{j+2}}{\lambda}\mathbf{v}_{j+2} - \cdots - \frac{\mu_{p+s}}{\lambda}\mathbf{v}_{p+s}\right) \\ &\quad\quad +\delta_{j+2}\mathbf{v}_{j+2} + \cdots + \delta_{p+s}\mathbf{v}_{p+s} \\ &= \left(\delta_1 - \frac{\delta_{j+1}\kappa_1}{\lambda}\right)\mathbf{u}_1 + \cdots + \left(\delta_j - \frac{\delta_{j+1}\kappa_j}{\lambda}\right)\mathbf{u}_j + \frac{\delta_{j+1}}{\lambda}\mathbf{u}_{j+1} \\ &\quad +\left(\delta_{j+2} - \frac{\delta_{j+1}\mu_{j+2}}{\lambda}\right)\mathbf{v}_{j+2} + \cdots + \left(\delta_{p+s} - \frac{\delta_{j+1}\mu_{p+s}}{\lambda}\right)\mathbf{v}_{p+s}\end{aligned}$$

It follows that $\mathbf{u}_1, \cdots, \mathbf{u}_j, \mathbf{u}_{j+1}, \mathbf{v}_{j+2} \cdots, \mathbf{v}_{p+s}$ constitute a basis for $\mathcal{W}$.

(c) Hence inductively, we deduce that, after relabelling the indices of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p, \mathbf{v}_{p+1}, \cdots, \mathbf{v}_{p+s}$ if necessary, we obtain some basis for $\mathcal{W}$ over $\mathbb{R}$, given by $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \cdots, \mathbf{u}_p, \mathbf{v}_{p+1}, \cdots, \mathbf{v}_{p+s}$.

9. We now complete the argument for the Replacement Theorem.

Proof of Theorem (7).

Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$ over the reals. Let p, q be positive integers.

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p, \mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q \in \mathcal{W}$.

Suppose $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$ constitute a basis for $\mathcal{W}$ over the reals.

Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are linearly independent over the reals.

(a) We verify the inequality $p \leq q$, with the help of the method of proof-by-contradiction:

- Suppose it were true that $p > q$. (Then $p \geq q + 1$.)
  [We focus on $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q, \mathbf{u}_{q+1}$.]
  Since $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ are just q column vectors amongst $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$, it would happen that the column vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ would be linearly independent over the reals.
  Now, by Theorem (7"), we would obtain a basis for $\mathcal{W}$ over the reals with $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ (and some $q-q = 0$ column vectors from amongst $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$).
  Since $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ constituted a basis for $\mathcal{W}$ over the reals and $\mathbf{u}_{q+1}$ belongs to $\mathcal{W}$, it would happen that $\mathbf{u}_{q+1}$ would be a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ over the reals.
  But this is impossible because $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q, \mathbf{u}_{q+1}$ are linearly independent over the reals.

(b) Now we have verified that $p \leq q$. Write $s = q - p$.
According to Theorem (7"), $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$, and some s column vectors amongst $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{p+s}$ together, constitute a basis of $\mathcal{W}$ over the reals.

10. We now apply the Replacement Theorem to prove several results.

Theorem (3). (Uniqueness of 'size' of various bases of the same subspace of $\mathbb{R}^n$ over the reals.)

Any two bases for a subspace of $\mathbb{R}^n$ over the reals have the same number of column vectors.

Theorem (6).

$\mathbb{R}^n$ is the only n-dimensional subspace of $\mathbb{R}^n$ over the reals.

Theorem (8).

Let $\mathcal{V}$ be a q-dimensional subspace of $\mathbb{R}^n$ over the reals.

The statements below hold:—

(a) *Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_\ell \in \mathcal{V}$. Suppose $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_\ell$ are linearly independent over the reals.*
Then $\ell \leq q$.

(b) *For each positive integer k, any $q + k$ column vectors belonging to $\mathcal{V}$ are linearly dependent over the reals.*

Theorem (9).

Let $\mathcal{W}$ be a q-dimensional subspace of $\mathbb{R}^n$ over the reals.

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p \in \mathcal{W}$.

Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are linearly independent over the reals.

Then:—

(a) *the inequality $p \leq q$ holds, and*

(b) *there is some basis for $\mathcal{W}$ over real constituted by $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$, and some $q - p$ column vectors belonging to $\mathcal{W}$.*

11. **Proof of Theorem (3).**

[The idea in the argument is the same as that for Theorem (7).]

Let $\mathcal{V}$ be a subspace of $\mathbb{R}^n$ over the reals.

When $\mathcal{V} = \{\mathbf{0}_n\}$, the empty set is its one and only one basis, and there is nothing to prove here.

From now on we suppose $\mathcal{V}$ is not the zero subspace of $\mathbb{R}^n$ over $\mathbb{R}^n$.

Suppose $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_p$ constitute a basis for $\mathcal{V}$ over the reals.

Also suppose $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_{p'}$ also constitute a basis for $\mathcal{V}$ over the reals.

We verify that $p = p'$:—

- Suppose it were true that $p \neq p'$.
  Without loss of generality, assume $p < p'$. Note that $p + 1 \leq p'$.
  [We focus on $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_p, \mathbf{y}_{p+1}$.]
  By assumption, $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_p, \mathbf{y}_{p+1}, \cdots, \mathbf{y}_{p+s}$ constitute a basis for $\mathcal{V}$ over the reals.
  Then $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_p$ are linearly independent over the reals.
  Recall that by assumption $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_p$ constitute a basis for $\mathcal{V}$ over the reals.
  Therefore, by the Replacement Theorem, we obtain some basis for $\mathcal{V}$ over the reals, given by $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_p$, (and some $p - p = 0$ column vectors from amongst $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_p$).
  Since $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_p$ constitute a basis for $\mathcal{V}$ over the reals and $\mathbf{y}_{p+1}$ belongs to $\mathcal{V}$, the column $\mathbf{y}_{p+1}$ is a linear combination of $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_p$ over the reals.
  But this is impossible because $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_p, \mathbf{y}_{p+1}$ are linearly independent over the reals.
  Therefore it would be impossible for $p < p'$ to hold.
  [Modifying the argument above, we also show that it is impossible for $p' < p$ to hold.]
  Hence $p = p'$ in the first place.

12. **Proof of Theorem (6).**

We are going to prove the statement that reads:—

- '*If $\mathcal{W}$ is an n-dimensional subspace of $\mathbb{R}^n$ over the reals then $\mathcal{W} = \mathbb{R}^n$.*'

Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$ over the reals. Suppose that $\dim(\mathcal{W}) = n$.

By definition, there is some basis for $\mathcal{W}$ over the reals with n column vectors with n real entries, say, $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$.

They are n linearly independent column vectors, belonging to $\mathbb{R}^n$.

Note that $\mathbf{e}_1^{(n)}, \mathbf{e}_2^{(n)}, \cdots, \mathbf{e}_n^{(n)}$ constitute some basis for $\mathbb{R}^n$.

Then by the Replacement Theorem, $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$, (and some $n - n = 0$ column vectors from amongst $\mathbf{e}_1^{(n)}, \mathbf{e}_2^{(n)}, \cdots, \mathbf{e}_n^{(n)}$,) constitute a basis for $\mathbb{R}^n$ over the reals.
It follows that $\mathbb{R}^n = \mathsf{Span}\,(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}) = \mathcal{W}$.

13. **Proof of Theorem (8).**

Let $\mathcal{V}$ be a q-dimensional subspace of $\mathbb{R}^n$ over the reals.

By assumption, we may pick some basis for $\mathcal{V}$ over the reals with q column vectors, say, $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$.

(a) Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_\ell \in \mathcal{V}$.
Suppose $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_\ell$ are linearly independent over the reals.
Then by the Replacement Theorem, $\ell \leq q$.

(b) What we have just verified (under the standing assumption that $\mathcal{V}$ *is a q-dimensional subspace of* $\mathbb{R}^n$ *over the reals*') is that:—

($\star$) *For any positive integer* ℓ, *for any* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_\ell \in \mathcal{V}$, *if* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_\ell$ *are linearly independent over the reals then* $\ell \leq q$.

By a purely logical consideration, we may re-formulate ($\star$) as:—

($\star'$) *For any positive integer* m, *for any* $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_m \in \mathcal{V}$, *if* $m > q$ *then* $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_m$ *are linearly dependent over the reals.*

The statement ($\star''$) is a re-formulation of ($\star'$):—

($\star''$) *For each positive integer* k, *any* $q + k$ *column vectors belonging to* $\mathcal{V}$ *are linearly dependent over the reals.*

14. **Proof of Theorem (9).**

Let $\mathcal{W}$ be a q-dimensional subspace of $\mathbb{R}^n$ over the reals.

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p \in \mathcal{W}$.

Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are linearly independent over the reals.

By assumption, we may pick some basis of $\mathcal{W}$ over the reals with q column vectors with n real entries, say, $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$.

Then, by the Replacement Theorem, it follows that:—

(a) the inequality $p \leq q$ holds, and

(b) there is some basis for $\mathcal{W}$ over real constituted by $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$, and some $q - p$ column vectors from amongst $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$, which belong to $\mathcal{W}$.