4.5 Dimension for subspaces of column vectors.

0. *Assumed background.*

   - Whatever has been covered in Topics 1-3, especially:—
     - ∗ 1.5 *Linear combinations.*
     - ∗ 1.6 *Linear dependence and linear independence.*
   - 4.1 *Sets of matrices and sets of vectors.*
   - 4.2 *Set equality (for sets of matrices and sets of vectors).*
   - 4.3 *Subspaces of column vectors.*
   - 4.4 *Basis for subspaces of column vectors.*

   Abstract. We introduce:—

   - the dimension for an arbitrary subspace of $\mathbb{R}^n$,
   - the Replacement Theorem and its consequences.

   In the *appendix*, we provide the proofs of the fundamental results about basis and dimension for subspaces of $\mathbb{R}^p$ that we are going to state.

1. **Questions about the notion of basis for a general subspace of $\mathbb{R}^n$.**

   We can (and should) ask several questions about the notion of basis for a general subspace, say, $\mathcal{V}$, of $\mathbb{R}^n$ over the reals.

   Question (1).

   Is it guaranteed that $\mathcal{V}$ has a basis over the reals?

   Answer.

   (a) When $\mathcal{V} = \mathbb{R}^n$, the answer is *yes.* But what if $\mathcal{V} \neq \mathbb{R}^n$?

   (b) If $\mathcal{V}$ is itself the span of some k linearly independent column vectors, say, $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$, over the reals, then, by definition, these k column vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$, certainly constitute a basis for $\mathcal{V}$ over the reals.

   But what if $\mathcal{V}$ is not already known to be the span of some linearly independent column vectors over the reals?

   Question (2). *If $\mathcal{V}$ has a basis over the reals, how many bases does $\mathcal{V}$ have?*

   Answer.
   When $\mathcal{V} = \mathbb{R}^n$, the answer is definitely *no.*

   This suggests the answer to this question is *many more than one*, when $\mathcal{V}$ is not the zero subspace of $\mathbb{R}^n$.

   In fact, if $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ constitute a basis of $\mathcal{V}$ over the reals, then for each choice of non-zero real numbers $\alpha_1, \alpha_2, \cdots, \alpha_p$, we have a distinct basis of $\mathcal{V}$ over the reals, given by $\alpha_1\mathbf{u}_1, \alpha_2\mathbf{u}_2, \cdots, \alpha_p\mathbf{u}_p$.

2. In view of the answer to Question (2), we should ask these further questions:—

   Question (3).

   Is is possible to compare various bases of $\mathcal{V}$ over the reals? How can the comparison be done? Can it be done systematically?

   Question (4).

   Is there anything common amongst all bases of the same $\mathcal{V}$ over the reals?

   The 'Change-of-basis' Theorem has provided a partial answer to Question (3). But it is relevant only under the assumption that various bases for $\mathcal{V}$ have the same number of column vectors. Now this becomes a matter of concern of Question (4).

   We provide the answers to these questions in the theoretical results below.

3. We start by recalling two things about linear independence for column vectors with n real entries:—

   (a) *No $n + 1$ or more column vectors with n real entries are linearly independent over the reals.*

   (b) *By definition of the notion of basis, every basis for a subspace of $\mathbb{R}^n$ over the reals is necessarily made up of linearly independent column vectors with n real entries.*

Combining these two observations, we have the result below. In plain words, this says that the 'size' of a basis for a subspace of $\mathbb{R}^n$ 'cannot be too large'.

Theorem (1). (Upper bound of number of column vectors in a basis for a subspace of $\mathbb{R}^n$ over the reals.)

Every basis for any subspace of $\mathbb{R}^n$ over the reals has at most n column vectors.

4. Theorem (1) can be much sharpened to give the key theoretical result below. This result provides an answer to Question (1).

   Theorem (2). (Existence of basis for an arbitrary non-zero subspace of $\mathbb{R}^n$ over the reals.)

   Suppose $\mathcal{V}$ is a non-zero subspace of $\mathbb{R}^n$ over the reals. Then there is a basis for $\mathcal{V}$ over the reals which consists of at least one and at most n column vectors with n real entries.

   Remark. The proof of Theorem (2) is provided in the *appendix.*

   Further remark. Theorem (2) does not have any practical application within the scope of this course. Its only purpose is to guarantee that we can talk about basis for an arbitrary subspace of $\mathbb{R}^n$.

5. We now provide some kind of answer to Question (4).

   Theorem (3). (Uniqueness of 'size' of various bases of the same subspace of $\mathbb{R}^n$ over the reals.)

   Any two bases for a subspace of $\mathbb{R}^n$ over the reals have the same number of column vectors.

   Remark. The proof of Theorem (3) is provided in the *appendix.*

   Further remark. The argument will heavily rely on a theoretical device, known as the **Replacement Theorem**, whose statement will be introduced later. The Replacement Theorem can be regarded as the ultimate answer to Question (3).

6. In the light of the validity of Theorem (2) and Theorem (3), it makes sense to introduce the notion of dimension of subspace below. In many situation it is what we really need from the consideration of subspaces.

   Definition. (Dimension of a subspace of $\mathbb{R}^n$ over the reals.)

   Let $\mathcal{V}$ be a subspace of $\mathbb{R}^n$ over the reals.

   (1) *(Suppose $\mathcal{V}$ is the zero subspace of $\mathbb{R}^n$ over the reals.) We declare the dimension of $\{\mathbf{0}_n\}$ over the reals is* 0, *and write* $\dim_{\mathbb{R}}(\{\mathbf{0}_p\}) = 0$.

   (2) *Suppose $\mathcal{V}$ is not the zero subspace of $\mathbb{R}^n$ over the reals.*
   We call the number of column vectors belonging to a basis for $\mathcal{V}$ over the reals the **dimension of $\mathcal{V}$ over the reals.**
   When such a number is p, we write $\dim_{\mathbb{R}}(\mathcal{V}) = p$, and we refer to $\mathcal{V}$ as a p-dimensional subspace of $\mathbb{R}^n$ over the reals.

7. **Comments on the definition for the notion of dimension.**

   (a) According to Theorem (2), it makes sense to talk about the number of column vectors belonging to a basis for $\mathcal{V}$ over $\mathbb{R}$, because such a basis exists.

   (b) According to Theorem (3), it makes sense to refer to such a number as something determined by $\mathcal{V}$ (and introduce the notation $\dim_{\mathbb{R}}(\mathcal{V})$), because the numbers of various bases for the same $\mathcal{V}$ over the reals are the same.

   (c) According to Theorem (2) again, we know that $\dim_{\mathbb{R}}(\mathcal{V})$ is an integer between 0 and n.

   (d) We may simplify the notation $\dim_{\mathbb{R}}(\mathcal{V})$ as $\dim(\mathcal{V})$, with the understanding that only subspaces over the reals are involved in the discussion.

8. Immediately following from the definitions for the notions of basis and dimension is the result below about span.

   Theorem (4).

   Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p \in \mathbb{R}^n$.

   Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are linearly independent over the reals.

   Then:—

   (a) *the inequality $p \leq n$ holds, and*

   (b) $\mathrm{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p\})$ *is a p-dimensional subspace of $\mathbb{R}^n$ over the reals, with a basis over the reals constituted by $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$.*

9. **Example (1). (Easy illustration of Theorem (4).)**

Consider the standard base $\mathbf{e}_1^{(n)}, \mathbf{e}_2^{(n)}, \cdots, \mathbf{e}_n^{(n)}$ for $\mathbb{R}^n$.

For each $q = 1, 2, \cdots, n$, the q column vectors $\mathbf{e}_1^{(n)}, \mathbf{e}_2^{(n)}, \cdots, \mathbf{e}_q^{(n)}$ constitute a basis for the subspace of $\mathbb{R}^n$ which is

$$\text{Span}(\{\mathbf{e}_1^{(n)}, \mathbf{e}_2^{(n)}, \cdots, \mathbf{e}_q^{(n)}\}).$$

This subspace of $\mathbb{R}^n$ is of dimension q.

10. **Example (2). (Illustration on Theorem (4).)**

(a) *Let* $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, *and* $\mathcal{W} = \text{Span}(\{\mathbf{u}_1, \mathbf{u}_2\})$.

We can verify that $\mathbf{u}_1, \mathbf{u}_2$ are linearly independent over the reals.
Therefore $\mathbf{u}_1, \mathbf{u}_2$ constitute a basis for $\mathcal{W}$ over the reals.
Hence $\dim(\mathcal{W}) = 2$.

(b) *Let* $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$, *and* $\mathcal{W} = \text{Span}(\{\mathbf{u}_1, \mathbf{u}_2\})$.

We can verify that $\mathbf{u}_1, \mathbf{u}_2$ are linearly independent over the reals.
Therefore $\mathbf{u}_1, \mathbf{u}_2$ constitute a basis for $\mathcal{W}$ over the reals.
Hence $\dim(\mathcal{W}) = 2$.

(c) *Let* $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 0 \\ -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$, *and* $\mathcal{W} = \text{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\})$.

We can verify that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly independent over the reals.
Therefore $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ constitute a basis for $\mathcal{W}$ over the reals.
Hence $\dim(\mathcal{W}) = 3$.

11. **Example (3). (Further illustration on Theorem (4), through null spaces of matrices.)**

(a) *Let* $A = \begin{bmatrix} 1 & -1 & 2 & -7 & -23 \\ 3 & -2 & 6 & -18 & -55 \\ -4 & 3 & -7 & 23 & 73 \\ 1 & 2 & 0 & 7 & 33 \end{bmatrix}$.

We proceed to solve the homogeneous system $\mathcal{LS}(A, \mathbf{0}_4)$, by applying row operations to obtain a reduced row-echelon form which is row-equivalent to A:—

$$A \longrightarrow \cdots\cdots\cdots \longrightarrow A' = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}.$$

It follows that the null space $\mathcal{N}(A)$ of A is given by

$$\mathcal{N}(A) = \mathsf{Span}\ (\{\mathbf{u}_1\}),$$

in which $\mathbf{u}_1$ is the non-zero column vector (which is read off from A' and) which is given by

$$\mathbf{u}_1 = \begin{bmatrix} -1 \\ -2 \\ -3 \\ -4 \\ 1 \end{bmatrix}.$$

Note that $\mathbf{u}_1$ is linearly independent. Then $\mathbf{u}_1$ constitutes a basis for $\mathcal{N}(A)$ over the reals.
Therefore $\dim(\mathcal{N}(A)) = 1$.

(b) *Let* $A = \begin{bmatrix} 1 & 3 & 1 & -2 & 1 \\ 1 & 3 & 2 & -3 & -3 \\ 2 & 6 & 1 & -2 & 10 \\ -1 & -3 & -3 & 1 & -5 \end{bmatrix}$.

We proceed to solve the homogeneous system $\mathcal{LS}(A, \mathbf{0}_4)$, by applying row operations to obtain a reduced row-echelon form which is row-equivalent to A:—

$$A \longrightarrow \cdots\cdots\cdots \longrightarrow A' = \begin{bmatrix} 1 & 3 & 0 & 0 & 9 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It follows that the null space $\mathcal{N}(A)$ of A is given by

$$\mathcal{N}(A) = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2\}),$$

in which $\mathbf{u}_1, \mathbf{u}_2$ are the linearly independent column vectors (which are read off from A' and) which are given by

$$\mathbf{u}_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} -9 \\ 0 \\ 0 \\ -4 \\ 1 \end{bmatrix}.$$

Then $\mathbf{u}_1, \mathbf{u}_2$ constitute a basis for $\mathcal{N}(A)$ over the reals.

Therefore $\dim(\mathcal{N}(A)) = 2$.

(c) *Let* $A = \begin{bmatrix} 0 & 0 & 2 & 3 & 5 & -7 \\ -1 & 2 & 1 & -1 & 0 & -2 \\ 2 & -4 & -1 & 3 & 2 & 1 \\ 3 & -6 & -1 & 5 & 4 & 0 \end{bmatrix}$.

We proceed to solve the homogeneous system $\mathcal{LS}(A, \mathbf{0}_4)$, by applying row operations to obtain a reduced row-echelon form which is row-equivalent to A:—

$$A \longrightarrow \cdots\cdots\cdots \longrightarrow A' = \begin{bmatrix} 1 & -2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It follows that the null space $\mathcal{N}(A)$ of A is given by

$$\mathcal{N}(A) = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}),$$

in which $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are the linearly independent column vectors (which are read off from A' and) which are given by

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ constitute a basis for $\mathcal{N}(A)$ over the reals.

Therefore $\dim(\mathcal{N}(A)) = 3$.

12. A special (and extreme) case in Example (1) deserves to be singled out and stated as a theorem.

Theorem (5).

$\mathbb{R}^n$ is an n-dimensional subspace of $\mathbb{R}^n$ over the reals.

Proof of Theorem (5).

The n column vectors $\mathbf{e}_1^{(n)}, \mathbf{e}_2^{(n)}, \cdots, \mathbf{e}_n^{(n)}$ constitute a basis for $\mathbb{R}^n$ over the reals.

13. We can sharpen the result much more:—

Theorem (6).

$\mathbb{R}^n$ is the only n-dimensional subspace of $\mathbb{R}^n$ over the reals.

Remark. The proof of this result is given in the *appendix*. The argument relies on the Replacement Theorem, to be introduced immediately.

14. **Theorem (7). (Replacement Theorem.)**

Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$ over the reals. Let p, q be positive integers.

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p, \mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q \in \mathcal{W}$.

Suppose $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$ constitute a basis for $\mathcal{W}$ over the reals.

Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are linearly independent over the reals.

Then:—

(a) *the inequality $p \leq q$ holds, and*

(b) *$\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$, and some $q-p$ column vectors amongst $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$ together, constitute a basis of $\mathcal{W}$ over the reals.*

15. We state two theoretical results which follows immediately from the Replacement Theorem. Their proofs are provided in the *appendix*.

The first of them is a generalization of the result below that says:—

> *any $p+1$ or more column vectors belonging to $\mathbb{R}^p$ definitely linearly dependent over the reals.*

Theorem (8). (Corollary (1) to Theorem (7).)

Let $\mathcal{V}$ be a q-dimensional subspace of $\mathbb{R}^n$ over the reals.

The statements below hold:—

(a) *Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_\ell \in \mathcal{V}$. Suppose $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_\ell$ are linearly independent over the reals.*
Then $\ell \leq q$.

(b) *For each positive integer k, any $q+k$ column vectors belonging to $\mathcal{V}$ are linearly dependent over the reals.*

Remark. In plain words, each part of Theorem (8) says that:—

> *any $q+1$ or more column vectors belonging to a q-dimensional subspace of $\mathbb{R}^n$ over the reals are definitely linearly dependent over the reals.*

16. Here comes another result that follows from the Replacement Theorem.

Theorem (9). (Corollary (2) to Theorem (7).)

Let $\mathcal{W}$ be a q-dimensional subspace of $\mathbb{R}^n$ over the reals.

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p \in \mathcal{W}$.

Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are linearly independent over the reals.

Then:—

(a) *the inequality $p \leq q$ holds, and*

(b) *there is some basis for $\mathcal{W}$ over real constituted by $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$, and some $q-p$ column vectors belonging to $\mathcal{W}$.*

Remark. This is how part (b) of the conclusion is very often interpreted:—

> *Given any collection of p linearly independent column vectors in a q-dimensional subspace $\mathcal{W}$ of $\mathbb{R}^n$ over the reals, it is possible to 'extend' this collection to a basis for $\mathcal{W}$ over the reals.*

17. The example below partially illustrates the content of the Replacement Theorem, and also suggests how the 'algebra' involved in its argument is done. Viewed in another way, this example also illustrates the idea in Theorem (9).

Example (4). (Illustration on the content of the Replacement Theorem, and the idea in Theorem (9).)

Regard $\mathbb{R}^8$ as a subspace of $\mathbb{R}^8$ itself over the reals.

We have the standard base for $\mathbb{R}^8$, constituted by $\mathbf{e}_1^{(8)}, \mathbf{e}_2^{(8)}, \mathbf{e}_3^{(8)}, \mathbf{e}_4^{(8)}, \mathbf{e}_5^{(8)}, \mathbf{e}_6^{(8)}, \mathbf{e}_7^{(8)}, \mathbf{e}_8^{(8)}$.

For simplicity, we write $\mathbf{e}_j^{(8)}$ as $\mathbf{e}_j$ for each j.

Let $\mathbf{u}_1 = \mathbf{e}_1 + \mathbf{e}_2$, $\mathbf{u}_2 = \mathbf{e}_2 + \mathbf{e}_3$, $\mathbf{u}_3 = \mathbf{e}_1 + \mathbf{e}_3$, $\mathbf{u}_4 = \mathbf{e}_4 + \mathbf{e}_5$, $\mathbf{u}_5 = -\mathbf{e}_4 + \mathbf{e}_5$, $\mathbf{u}_6 = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5 + \mathbf{e}_6$.

(a) Note that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6$ are linearly independent over $\mathbb{R}^6$. (Fill in the detail as exercise.)

(b) At a theoretical level, we know from the Replacement Theorem that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6$, and 2 appropriate column vectors from amongst $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7, \mathbf{e}_8$ together, constitute a basis for $\mathbb{R}^8$ over the reals.

(c) Another way of saying the same thing (but with emphasis shifted) is that we may 'extend' $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6$ to give a basis for $\mathbb{R}^n$ over the reals, by incorporating 2 column vectors from amongst $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7, \mathbf{e}_8$.

(d) In practice, for the selection to be done, we need to do some algebra about linear combinations.

The strategy is to obtain a desired basis for $\mathbb{R}^8$ from the original basis for $\mathbb{R}^8$ by constructing various bases for $\mathbb{R}^8$ with more and more $\mathbf{u}_k$'s, and fewer and fewer $\mathbf{e}_\ell$'s. At each step, we replace one of the remaining $\mathbf{e}_\ell$'s in the 'intermediate' basis with an appropriate $\mathbf{u}_k$ which is yet to be incorporated.

i. By definition $\mathbf{u}_1 = \mathbf{e}_1 + \mathbf{e}_2$. —— $(\star_1)$
Then $\mathbf{e}_1 = \mathbf{u}_1 - \mathbf{e}_2$. —— $(\star_1')$
Using the equalities $(\star_1), (\star_1')$, we deduce that $\mathbf{u}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7, \mathbf{e}_8$ constitute a basis for $\mathbb{R}^8$ over the reals.

ii. By definition $\mathbf{u}_2 = \mathbf{e}_2 + \mathbf{e}_3$. —— $(\star_2)$
Then $\mathbf{e}_2 = \mathbf{u}_2 - \mathbf{e}_3$. —— $(\star_2')$
Using the equalities $(\star_2), (\star_2')$, we deduce that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7, \mathbf{e}_8$ constitute a basis for $\mathbb{R}^8$ over the reals.

iii. By definition $\mathbf{u}_3 = \mathbf{e}_1 + \mathbf{e}_3$.
Note that $\mathbf{e}_1 = \mathbf{u}_1 - \mathbf{e}_2 = \mathbf{u}_1 - (\mathbf{u}_2 - \mathbf{e}_3) = \mathbf{u}_1 - \mathbf{u}_2 + \mathbf{e}_3$.
Therefore $\mathbf{u}_3 = \mathbf{u}_1 - \mathbf{u}_2 + 2\mathbf{e}_3$. —— $(\star_2)$
Then $\mathbf{e}_3 = -\frac{1}{2}\mathbf{u}_1 + \frac{1}{2}\mathbf{u}_2 + \frac{1}{2}\mathbf{u}_3$. —— $(\star_3')$
Using the equalities $(\star_3), (\star_3')$, we deduce that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7, \mathbf{e}_8$ constitute a basis for $\mathbb{R}^8$ over the reals.

iv. By definition $\mathbf{u}_4 = \mathbf{e}_4 + \mathbf{e}_5$. —— $(\star_4)$
Then $\mathbf{e}_4 = \mathbf{u}_4 - \mathbf{e}_5$. —— $(\star_4')$
Using the equalities $(\star_4), (\star_4')$, we deduce that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7, \mathbf{e}_8$ constitute a basis for $\mathbb{R}^8$ over the reals.

v. By definition $\mathbf{u}_5 = -\mathbf{e}_4 + \mathbf{e}_5$.
Note that $\mathbf{e}_4 = \mathbf{u}_4 - \mathbf{e}_5$.
Therefore $\mathbf{u}_5 = -\mathbf{u}_4 + 2\mathbf{e}_5$.—— $(\star_5)$
Then $\mathbf{e}_5 = \frac{1}{2}\mathbf{u}_4 + \frac{1}{2}\mathbf{u}_5$. —— $(\star_5')$
Using the equalities $(\star_5), (\star_5')$, we deduce that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{e}_6, \mathbf{e}_7, \mathbf{e}_8$ constitute a basis for $\mathbb{R}^8$ over the reals.

vi. By definition $\mathbf{u}_6 = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5 + \mathbf{e}_6$.
After some methodical application (or some clever observation), we deduce $\mathbf{u}_6 = \frac{1}{2}\mathbf{u}_1 + \frac{1}{2}\mathbf{u}_2 + \frac{1}{2}\mathbf{u}_3 + \mathbf{u}_4 + \mathbf{e}_6$. —— $(\star_6)$
Then $\mathbf{e}_6 = -\frac{1}{2}\mathbf{u}_1 - \frac{1}{2}\mathbf{u}_2 - \frac{1}{2}\mathbf{u}_3 - \mathbf{u}_4 + \mathbf{u}_6$. —— $(\star_6)$
Using the equalities $(\star_6), (\star_6')$, we deduce that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6, \mathbf{e}_7, \mathbf{e}_8$ constitute a basis for $\mathbb{R}^8$ over the reals.