4.4.1 Appendix: Proof of the 'Change-of-basis' Theorem.

0. The material in this appendix is supplementary.

1. We are going to prove the theoretical result introduced as:—

   Theorem (4). ('Change-of-basis' Theorem.)

   Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$ over the reals.

   Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ constitute a basis for $\mathcal{W}$ over the reals. Also suppose $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$ constitute a basis for $\mathcal{W}$ over the reals.

   Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_p\ ]$, $V = [\ \mathbf{v}_1 \mid \mathbf{v}_2 \mid \cdots \mid \mathbf{v}_p\ ]$.

   Then the statements below hold:—

   (a) *There exists some unique invertible $(p \times p)$-square matrix S with real entries such that $U = VS$.*

   (b) *Let $\mathbf{x} \in \mathcal{W}$, and $\alpha_1, \alpha_2, \cdots, \alpha_p, \beta_1, \beta_2, \cdots, \beta_p \in \mathbb{R}$.*
   Suppose $\mathbf{x} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_p\mathbf{u}_p$ and $\mathbf{x} = \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \cdots + \beta_p\mathbf{v}_p$

   Then $\begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{bmatrix} = S \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_p \end{bmatrix}$, *for the same invertible $(p \times p)$-square matrix S above.*

   Remark on terminologies. The matrix S, which relates the matrices U, V via the equality $U = VS$, is called the **change-of-basis matrix relating the (ordered) basis $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ for $\mathcal{W}$ to the (ordered) basis $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$ for $\mathcal{W}$.**

2. In the argument for Theorem (4), we will use the result below which links up the notions of linear dependence and linear independence with homogeneous systems of linear equations:—

   Lemma ($\sharp$).

   Suppose $\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_p$ are column vectors with n real entries, and Z is the $(n \times p)$-matrix with real entries given by $Z = [\ \mathbf{z}_1 \mid \mathbf{z}_2 \mid \cdots \mid \mathbf{z}_p\ ]$.

   Then:—

   (a) *The statements (LD), (LD_0) are logically equivalent:—*

   (LD) *$\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_p$ are linearly dependent over the reals.*

   (LD_0) *The homogeneous system $\mathcal{LS}(Z, \mathbf{0}_n)$ has some non-trivial solution.*

   (b) *The statements (LI), (LI_0) are logically equivalent:—*

   (LI) *$\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_p$ are linearly independent over the reals.*

   (LI_0) *The homogeneous system $\mathcal{LS}(Z, \mathbf{0}_n)$ has no non-trivial solution.*

3. **Proof of Theorem (4).**

   Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$.

   Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ constitute a basis for $\mathcal{W}$.

   Also suppose $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$ constitute a basis for $\mathcal{W}$.

   Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_p\ ]$, $V = [\ \mathbf{v}_1 \mid \mathbf{v}_2 \mid \cdots \mid \mathbf{v}_p\ ]$.

   (a) [We are going to show that there exists some invertible $(p \times p)$-square matrix S with real entries such that $U = VS$.]

      i. [We want to write down some matrix S for which the equality $U = VS$ holds.
      Note that for such an equality to make sense, it is necessary for S to be a $(p \times p)$-square matrix in the first place.]
      (Recall that $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$ constitute a basis for $\mathcal{W}$ over the reals.)
      For each $k = 1, 2, \cdots, p$, since $\mathbf{u}_k \in \mathcal{W}$, there exist some $s_{1k}, s_{2k}, \cdots, s_{pk} \in \mathbb{R}$ such that $\mathbf{u}_k = s_{1k}\mathbf{v}_1 + s_{2k}\mathbf{v}_2 + \cdots + s_{pk}\mathbf{v}_{pk}$.
      We write $\mathbf{s}_k = \begin{bmatrix} s_{1k} \\ s_{2k} \\ \vdots \\ s_{pk} \end{bmatrix}$. Then we have $\mathbf{u}_k = [\ \mathbf{v}_1 \mid \mathbf{v}_2 \mid \cdots \mid \mathbf{v}_p\ ] \begin{bmatrix} s_{1k} \\ s_{2k} \\ \vdots \\ s_{pk} \end{bmatrix} = V\mathbf{s}_k$.
      Now define $S = [\ \mathbf{s}_1 \mid \mathbf{s}_2 \mid \cdots \mid \mathbf{s}_p\ ]$. Note that S is a $(p \times p)$-square matrix with real entries.
      Therefore $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_p\ ] = [\ V\mathbf{s}_1 \mid V\mathbf{s}_2 \mid \cdots \mid V\mathbf{s}_p\ ] = V[\ \mathbf{s}_1 \mid \mathbf{s}_2 \mid \cdots \mid \mathbf{s}_p\ ] = VS$.

ii. [We want to write down some matrix T for which the equality $V = UT$ holds.]
(Recall that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ constitute a basis for $\mathcal{W}$ over the reals.)
For each $\ell = 1, 2, \cdots, p$, since $\mathbf{v}_\ell \in \mathcal{W}$, there exist some $t_{1\ell}, t_{2\ell}, \cdots, t_{p\ell} \in \mathbb{R}$ such that $\mathbf{v}_\ell = t_{1\ell}\mathbf{u}_1 + t_{2\ell}\mathbf{u}_2 + \cdots + t_{p\ell}\mathbf{u}_{p\ell}$.
We write $\mathbf{t}_\ell = \begin{bmatrix} t_{1\ell} \\ t_{2\ell} \\ \vdots \\ t_{p\ell} \end{bmatrix}$. Repeating the argument above, we also deduce that $\mathbf{v}_\ell = U\mathbf{t}_\ell$.
We define $T = [\ \mathbf{t}_1 \mid \mathbf{t}_2 \mid \cdots \mid \mathbf{t}_p\ ]$. Note that T is a $(p \times p)$-square matrix with real entries.
Further repeating the argument above, we also deduce that $V = UT$.

iii. [We want to verify that $TS = I_p$.]
We have the equalities $U = VS$ and $V = UT$.
Then $U = VS = (UT)S = U(TS)$.
Since S, T are $(p \times p)$-square matrices, TS is also a $(p \times p)$-square matrix.
For each $i, j = 1, 2, \cdots, p$, denote the j-th column of TS by $\mathbf{g}_j$.
[We want to verify that $\mathbf{g}_j = \mathbf{e}_j^{(p)}$.]
Note that $U(I_p - TS) = U - UTS = \mathcal{O}_{n\times p}$.
Then, for each $j = 1, 2, \cdots, p$, the j-th column of $I_p - TS$ is $\mathbf{e}_j^{(p)} - \mathbf{g}_j$.
Then we have $U(\mathbf{e}_j^{(p)} - \mathbf{g}_j) = \mathbf{0}_n$. Therefore $\mathbf{e}_j^{(p)} - \mathbf{g}_j$ is a solution of the homogeneous system $\mathcal{LS}(U,\ \mathbf{0}_n)$.
Since $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ constitute a basis for $\mathcal{W}$ over the reals, $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are linearly independent over the reals.
Then, by Lemma ($\sharp$), the homogeneous system $\mathcal{LS}(U,\ \mathbf{0}_n)$ has no non-trivial solution.
Therefore $\mathbf{e}_j^{(p)} - \mathbf{g}_j = \mathbf{0}_n$. Hence $\mathbf{g}_j = \mathbf{e}_j^{(p)}$.
It follows that $TS = I_p$.

iv. Repeating the argument above, but starting with the equalities $V = UT = (VS)T = V(ST)$, we also deduce the equality $ST = I_p$ (from the assumption that $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$ constitute a basis for $\mathcal{W}$ over the reals, and with the help of Lemma ($\sharp$)).
It follows that S, T are invertible, and they are matrix inverses of each other.

v. [We want to verify that S is the only invertible $(p\times p)$-square matrix for which the equality $U = VS$ holds.]
Let $\widetilde{S}$ be an invertible $(p \times p)$-square matrix. Suppose the equality $U = V\widetilde{S}$ holds.
For each $i, j = 1, 2, \cdots, p$, denote the j-th column of $\widetilde{S}$ by $\widetilde{\mathbf{s}}_j$, and the (i, j)-th entry of $\widetilde{S}$ by $\widetilde{s}_{ij}$.
[We want to verify that $\widetilde{\mathbf{s}}_j = \mathbf{s}_j$ for each $j = 1, 2, \cdots, p$.]
We have the equalities $U = VS$ and $U = V\widetilde{S}$. Then $V(S - \widetilde{S}) = \mathcal{O}_{n\times p}$.
For each $j = 1, 2, \cdots, p$, the j-th column of $S - \widetilde{S}$ is $\mathbf{s}_j - \widetilde{s}_j$.
Then $V(\mathbf{s}_j - \widetilde{s}_j) = \mathbf{0}_n$. Therefore $\mathbf{s}_j - \widetilde{s}_j$ is a solution of the homogeneous system $\mathcal{LS}(V,\ \mathbf{0}_n)$.
Since $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$ constitute a basis for $\mathcal{W}$ over the reals, $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$ are linearly independent over the reals.
Then, by Lemma ($\sharp$), the homogeneous system $\mathcal{LS}(V,\ \mathbf{0}_n)$ has no non-trivial solution.
Therefore $\mathbf{s}_j - \widetilde{s}_j = \mathbf{0}_n$. Hence $\widetilde{s}_j = \mathbf{s}_j$.
It follows that $\widetilde{S} = S$.

(b) Let $\mathbf{x} \in \mathcal{W}$, and $\alpha_1, \alpha_2, \cdots, \alpha_p, \beta_1, \beta_2, \cdots, \beta_p \in \mathbb{R}$.
Suppose $\mathbf{x} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_p\mathbf{u}_p$ and $\mathbf{x} = \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \cdots + \beta_p\mathbf{v}_p$.
Write $\mathbf{a} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_p \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{bmatrix}$.
We have $\mathbf{x} = U\mathbf{a}$ and $\mathbf{x} = V\mathbf{b}$.
Then $V\mathbf{b} = \mathbf{x} = U\mathbf{a} = VS\mathbf{a}$.
Therefore $V(\mathbf{b} - S\mathbf{a}) = \mathbf{0}_n$.
Hence $\mathbf{b} - S\mathbf{a}$ is a solution of the homogeneous system $\mathcal{LS}(V,\ \mathbf{0}_n)$.
Again using the linear independence of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$, and applying Lemma ($\sharp$) again, we deduce the equality $\mathbf{b} - S\mathbf{a} = \mathbf{0}_p$.
Then $\mathbf{b} = S\mathbf{a}$.