4.3 Subspaces of column vectors.

0. *Assumed background.*

   - Whatever has been covered in Topics 1-3.
   - 4.1 *Sets of matrices and sets of vectors.*
   - 4.2 *Set equality (for sets of matrices and sets of vectors).*

Abstract. We introduce:—

   - the notion of the vector space of all matrices of the same size,
   - the notion of subspace of the vector space of all column vectors with the same number of entries,
   - the notions of intersection and sum for subspaces.

1. **'Algebraic properties' of the set of all matrices of the same size.**

   Right from the beginning, we have known the validity of each statement in the list below about the properties of matrix addition and scalar multiplication. (Here they are slightly re-dressed with set notations.)

   (a) ii. *For any $A, B \in M_{p,q}(\mathbb{R})$, the equality $A + B = B + A$ holds.*
   iii. *For any $A, B, C \in M_{p,q}(\mathbb{R})$, the equality $(A + B) + C = A + (B + C)$ holds.*
   iv. *There exists some unique $Z \in M_{p,q}(\mathbb{R})$, namely, $Z = \mathcal{O}_{p\times q}$, such that for any $A \in M_{p,q}(\mathbb{R})$, the equalities $A + Z = A = Z + A$ hold.*
   v. *For any $A \in M_{p,q}(\mathbb{R})$, there exists some unique $Y \in M_{p,q}(\mathbb{R})$, namely, $Y = -A$, such that the equalities $A + Y = \mathcal{O}_{p\times q} = Y + A$ hold.*

   (b) ii. *For any $A \in M_{p,q}(\mathbb{R})$, for any $\alpha, \beta \in \mathbb{R}$, the equality $(\alpha\beta)A = \alpha(\beta A)$ holds.*
   iii. *For any $A \in M_{p,q}(\mathbb{R})$, the equality $1A = A$ holds.*

   (c) i. *For any $A \in M_{p,q}(\mathbb{R})$, for any $\alpha, \beta \in \mathbb{R}$, the equality $(\alpha + \beta)A = \alpha A + \beta A$ holds.*
   ii. *For any $A, B \in M_{p,q}(\mathbb{R})$, for any $\alpha \in \mathbb{R}$, the equality $\alpha(A + B) = \alpha A + \alpha B$ holds.*

   We now insert two extra statements in the list above.

   (a) i. *For any matrices A, B, if $A, B \in M_{p,q}(\mathbb{R})$ then $A + B \in M_{p,q}(\mathbb{R})$.*

   (b) i. *For any matrix A, for any $\alpha \in \mathbb{R}$, if $A \in M_{p,q}(\mathbb{R})$ then $\alpha A \in M_{p,q}(\mathbb{R})$.*

   These two statements look 'so obviously true' that we did not bother to state them earlier. In fact, their validity is 'embedded' in how matrix addition and scalar multiplication are defined.

   Because of the validity of the statements (a.i)-(a.v), (b.i)-(b.iii) and (c.i), (c.ii), $M_{p,q}(\mathbb{R})$ alongside matrix addition and scalar multiplication is called the **vector space of $(p \times q)$-matrices with real entries, over the reals**.

   A seemingly trivial consequence of the statements (a.i), (b.i) is the statement (d):—

   (d) *Suppose $A_1, A_2, \cdots, A_n \in M_{p,q}(\mathbb{R})$, and $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$. Then $\alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_n A_n \in M_{p,q}(\mathbb{R})$.*

   Or in plain words:

   > '*Every "linear combination over the reals" of (finitely many) $(p \times q)$-matrices with real entries will be also a $(p \times q)$-matrix with real entries.*'

2. In linear algebra we are interested in collections of matrices or vectors which possesses properties 'analogous' to the ones listed above for sets of all matrices of the same size. The 'analogy' is illustrated below:—

   Example (1). (Illustration, on how the 'algebraic properties' of the null space of an arbitrary matrix are 'analogous' to that of the set of all matrices of the same size.)

   Suppose H is a $(m \times p)$-matrix with real entries. Then the statements below hold:—

   (a) i. *For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^p$, if $\mathbf{u} \in \mathcal{N}(H)$ and $\mathbf{v} \in \mathcal{N}(H)$ then $\mathbf{u} + \mathbf{v} \in \mathcal{N}(H)$.*
   ii. *For any $\mathbf{u}, \mathbf{v} \in \mathcal{N}(H)$, the equality $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ holds.*
   iii. *For any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{N}(H)$, the equality $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ holds.*
   iv. *There exists some unique $\mathbf{z} \in \mathcal{N}(H)$, namely, $\mathbf{z} = \mathbf{0}_p$, such that for any $\mathbf{u} \in \mathcal{N}(H)$, the equalities $\mathbf{u} + \mathbf{z} = \mathbf{u} = \mathbf{z} + \mathbf{u}$ hold.*

v. *For any* $\mathbf{u} \in \mathcal{N}(H)$*, there exists some unique* $\mathbf{y} \in \mathcal{N}(H)$*, namely,* $\mathbf{y} = -\mathbf{u}$*, such that the equalities* $\mathbf{u} + \mathbf{y} = \mathbf{0}_p = \mathbf{y} + \mathbf{u}$ *hold.*

(b) i. *For any* $\mathbf{u} \in \mathbb{R}^p$*, for any* $\alpha \in \mathbb{R}$*, if* $\mathbf{u} \in \mathcal{N}(H)$ *then* $\alpha\mathbf{u} \in \mathcal{N}(H)$*.*

ii. *For any* $\mathbf{u} \in \mathcal{N}(H)$*, for any* $\alpha, \beta \in \mathbb{R}$*, the equality* $(\alpha\beta)\mathbf{u} = \alpha(\beta\mathbf{u})$ *holds.*

iii. *For any* $\mathbf{u} \in \mathcal{N}(H)$*, the equality* $1\mathbf{u} = \mathbf{u}$ *holds.*

(c) i. *For any* $\mathbf{u} \in \mathcal{N}(H)$*, for any* $\alpha, \beta \in \mathbb{R}$*, the equality* $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$ *holds.*

ii. *For any* $\mathbf{u}, \mathbf{v} \in \mathcal{N}(H)$*, for any* $\alpha \in \mathbb{R}$*, the equality* $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$ *holds.*

Moreover, the statement (d) *holds:—*

(d) *Suppose* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n \in \mathcal{N}(H)$*, and* $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$*. Then* $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n \in \mathcal{N}(H)$*.*

Justification:—

- The statements (a.ii), (a.iii), (b.ii), (b.iii), (c.i), (c.ii) are immediate consequences of the basic properties of vector addition and scalar multiplication.
- The statements (a.i), (a.iv), (a.v), (b.i) are respectively re-formulations of the statements (a.i'), (a.iv'), (a.v'), (b.i'), which are concerned with homogeneous systems:—
  - (a.i') *For any* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^p$*, if* $\mathbf{u}, \mathbf{v}$ *are solutions of* $\mathcal{LS}(H, \mathbf{0}_m)$ *then* $\mathbf{u} + \mathbf{v}$ *is a solution of* $\mathcal{LS}(H, \mathbf{0}_m)$*.*
  - (a.iv') *There exists some unique solution* $\mathbf{z}$ *of* $\mathcal{LS}(H, \mathbf{0}_m)$*, namely,* $\mathbf{z} = \mathbf{0}_p$*, such that for any solution* $\mathbf{u}$ *of* $\mathcal{LS}(H, \mathbf{0}_m)$*, the equalities* $\mathbf{u} + \mathbf{z} = \mathbf{u} = \mathbf{z} + \mathbf{u}$ *hold.*
  - (a.v') *For any solution* $\mathbf{u}$ *of* $\mathcal{LS}(H, \mathbf{0}_m)$*, there exists some unique solution* $\mathbf{y}$ *of* $\mathcal{LS}(H, \mathbf{0}_m)$*, namely,* $\mathbf{y} = -\mathbf{u}$*, such that the equalities* $\mathbf{u} + \mathbf{y} = \mathbf{0}_p = \mathbf{y} + \mathbf{u}$ *hold.*
  - (b.i') *For any* $\mathbf{u} \in \mathbb{R}^p$*, for any* $\alpha \in \mathbb{R}$*, if* $\mathbf{u}$ *is a solution of* $\mathcal{LS}(H, \mathbf{0}_m)$ *then* $\alpha\mathbf{u}$ *is a solution of* $\mathcal{LS}(H, \mathbf{0}_m)$*.*
- The statement (d) is a consequence of the statements (a.i), (b.i). An argument can be given with an application of mathematical induction.

3. **Comment on Example (1).**

The statements (a.i)-(a.v), (b.i)-(b.iii), (c.i), (c.ii), (d) describing the 'algebraic properties' of the null space of an arbitrary matrix can be formally 'obtained' from the corresponding statements when we consistently:—

- think of column vectors with the same number of entries instead of general matrices of the same size, and
- replace the symbol '$M_{p,q}(\mathbb{R})$' by the symbol '$\mathcal{N}(H)$'.

It is natural to for us to follow up by asking this question ($\star$):—

($\star$) *Given any set* T *of, say, column vectors with* p *real entries, can we simply replace the symbol '*$M_{p,q}(\mathbb{R})$*' by the symbol '*T*' consistently in the list of statements* (a.i)-(a.v), (b.i)-(b.iii), (c.i), (c.ii), (d) *on* $(p \times q)$*-matrices, so as to obtain a list of true statements on* T*?*

The answer to ($\star$) is *no*. The big uncertainty is concerned with the statements (a.i), (b.i) and a crucial part of the statement (a.iv). In general, there is no guarantee that the statements below are all true when we apply the 'replacement'.

(S1) $\mathbf{0}_p \in T$.

(S2) *For any* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^p$*, if* $\mathbf{u} \in T$ *and* $\mathbf{v} \in T$ *then* $\mathbf{u} + \mathbf{v} \in T$*.*

(S3) *For any* $\mathbf{u} \in \mathbb{R}^p$*, for any* $\alpha \in \mathbb{R}$*, if* $\mathbf{u} \in T$ *then* $\alpha\mathbf{u} \in T$*.*

As illustrated in Example (2), whether one or more of (S1), (S2), (S3) is true for such a set T of column vectors with, say, 2 entries will highly dependent on what T is.

4. **Example (2). (Illustrations on the negative answer to the question ($\star$).)**

(a) Suppose $T_1 = \left\{ \mathbf{x} \in \mathbb{R}^2 \;\middle|\; \text{Both entries of } \mathbf{x} \text{ are non-negative.} \right\}$.

The statement (S3) fails to hold for T_1. Justification:—

- We provide a counter-example against the statement (S3): '*For any* $\mathbf{u} \in \mathbb{R}^2$*, for any* $\alpha \in \mathbb{R}$*, if* $\mathbf{u} \in T_1$ *then* $\alpha\mathbf{u} \in T_1$*.*'

  Take $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Take $\alpha = -1$.

  By definition, $\mathbf{u} \in T_1$, and $\alpha \in \mathbb{R}$.

  Note that $\alpha\mathbf{u} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Then $\alpha\mathbf{u} \notin T_1$.

The statements (S1), (S2) holds for T_1. Justification:—

- Both entries of $\mathbf{0}_2$ are 0, and therefore are non-negative. Hence $\mathbf{0}_2 \in T_1$. (So (S1) holds.)
- We verify (S2): *For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, if $\mathbf{u} \in T_2$ and $\mathbf{v} \in T_2$ then $\mathbf{u} + \mathbf{v} \in T_2$.*
  Pick any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$. Suppose $\mathbf{u} \in T_2$ and $\mathbf{v} \in T_2$.
  Write $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Note that $\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$.
  By assumption, all of u_1, u_2, v_1, v_2 are non-negative.
  Then $u_1 + v_1$ and $u_2 + v_2$ are non-negative.
  Therefore $\mathbf{u} + \mathbf{v} \in T_1$.

(b) Suppose $T_2 = \left\{ \mathbf{x} \in \mathbb{R}^2 \;\middle|\; \text{At most one entry of } \mathbf{x} \text{ is non-zero.} \right\}$.

The statement (S2) fails to hold for T_2. Justification:—

- We provide a counter-example against the statement (S2): '*For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, if $\mathbf{u} \in T_2$ and $\mathbf{v} \in T_2$ then $\mathbf{u} + \mathbf{v} \in T_2$.*'
  Take $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
  By definition, $\mathbf{u} \in T_2$ and $\mathbf{v} \in T_2$.
  Note that $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then $\mathbf{u} + \mathbf{v} \notin T_2$.

The statements (S1), (S3) holds for T_2. Justification:—

- Both entries of $\mathbf{0}_2$ are 0. Hence $\mathbf{0}_2 \in T_2$. (So (S1) holds.)
- We verify (S3): *For any $\mathbf{u} \in \mathbb{R}^2$, for any $\alpha \in \mathbb{R}$, if $\mathbf{u} \in T_3$ then $\alpha\mathbf{u} \in T_3$.*
  Pick any $\mathbf{u} \in \mathbb{R}^2$. Pick any $\alpha \in \mathbb{R}$. Suppose $\mathbf{u} \in T_3$.
  Write $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$. Note that $\alpha\mathbf{u} = \begin{bmatrix} \alpha u_1 \\ \alpha u_2 \end{bmatrix}$.
  By assumption, at most one entry of $\mathbf{u}$ is non-zero. Then $u_1 = 0$ or $u_2 = 0$.
  Therefore $\alpha u_1 = 0$ or $\alpha u_2 = 0$. Hence at most one entry of $\alpha\mathbf{u}$ is non-zero.
  Then $\alpha\mathbf{u} \in T_2$.

(c) Suppose $T_3 = \emptyset$.

By assumption, $\mathbf{0}_2 \in T_3$. Hence the statement (S1) fails to hold for T_3.

By a purely logical consideration, the statements (S2), (S3) are true for T_3:—

(S2) *For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, if $\mathbf{u} \in T_3$ and $\mathbf{v} \in T_3$ then $\mathbf{u} + \mathbf{v} \in T_3$.*

(S3) *For any $\mathbf{u} \in \mathbb{R}^2$, for any $\alpha \in \mathbb{R}$, if $\mathbf{u} \in T_3$ then $\alpha\mathbf{u} \in T_3$.*

5. **Definition. (Subspaces of the vector spaces of column vectors with real entries.)**

   Let $\mathcal{W}$ be a set of column vectors with p real entries.

   We say that $\mathcal{W}$ **constitutes a subspace of $\mathbb{R}^p$ over the reals** *if and only if the statements* (S1), (S2), (S3) *hold simultaneously:—*

   (S1) $\mathbf{0}_p \in \mathcal{W}$.

   (S2) *For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^p$, if $\mathbf{u} \in \mathcal{W}$ and $\mathbf{v} \in \mathcal{W}$ then $\mathbf{u} + \mathbf{v} \in \mathcal{W}$.*

   (S3) *For any $\mathbf{u} \in \mathbb{R}^p$, for any $\alpha \in \mathbb{R}$, if $\mathbf{u} \in \mathcal{W}$ then $\alpha\mathbf{u} \in \mathcal{W}$.*

   Remark on terminology.

   (a) Some people reads (S2) as: '*Vector addition is closed in $\mathcal{W}$.*'

   (b) Some people reads (S3) as: '*Scalar multiplication is closed in $\mathcal{W}$.*'

6. An immediate consequence of the definition for the notion of subspaces (and the basic properties of vector addition and scalar multiplication for column vectors) is the result below:—

   Theorem (1).

   Suppose $\mathcal{W}$ is a subspace of $\mathbb{R}^p$ over the reals. Then the statements below hold:—

   (a) i. *For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^p$, if $\mathbf{u} \in \mathcal{W}$ and $\mathbf{v} \in \mathcal{W}$ then $\mathbf{u} + \mathbf{v} \in \mathcal{W}$.*
   ii. *For any $\mathbf{u}, \mathbf{v} \in \mathcal{W}$, the equality $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ holds.*
   iii. *For any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{W}$, the equality $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ holds.*
   iv. *There exists some unique $\mathbf{z} \in \mathcal{W}$, namely, $\mathbf{z} = \mathbf{0}_p$, such that for any $\mathbf{u} \in \mathcal{W}$, the equalities $\mathbf{u} + \mathbf{z} = \mathbf{u} = \mathbf{z} + \mathbf{u}$ hold.*

v. *For any* $\mathbf{u} \in \mathcal{W}$*, there exists some unique* $\mathbf{y} \in \mathcal{W}$*, namely,* $\mathbf{y} = -\mathbf{u}$*, such that the equalities* $\mathbf{u} + \mathbf{y} = \mathbf{0}_p = \mathbf{y} + \mathbf{u}$ *hold.*

(b) i. *For any* $\mathbf{u} \in \mathbb{R}^p$*, for any* $\alpha \in \mathbb{R}$*, if* $\mathbf{u} \in \mathcal{W}$ *then* $\alpha\mathbf{u} \in \mathcal{W}$*.*

ii. *For any* $\mathbf{u} \in \mathcal{W}$*, for any* $\alpha, \beta \in \mathbb{R}$*, the equality* $(\alpha\beta)\mathbf{u} = \alpha(\beta\mathbf{u})$ *holds.*

iii. *For any* $\mathbf{u} \in \mathcal{W}$*, the equality* $1\mathbf{u} = \mathbf{u}$ *holds.*

(c) i. *For any* $\mathbf{u} \in \mathcal{W}$*, for any* $\alpha, \beta \in \mathbb{R}$*, the equality* $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$ *holds.*

ii. *For any* $\mathbf{u}, \mathbf{v} \in \mathcal{W}$*, for any* $\alpha \in \mathbb{R}$*, the equality* $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$ *holds.*

7. Applying mathematical induction, and directly using Conditions (S2), (S3) in the definition for the notion of subspaces, we can prove the result below:—

   Lemma (2).

   Suppose $\mathcal{W}$ *is a subspace of* $\mathbb{R}^p$ *over the reals.*

   Then for any positive integer n*, if* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n \in \mathcal{W}$*, and* $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$*, then* $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n \in \mathcal{W}$*.*

   Remark. Put in plain words, Lemma (2) says:—

   > '*Every linear combination over the reals of (finitely many) column vectors belonging to a subspace of* $\mathbb{R}^p$ *will be also a column vector belonging to that same subspace of* $\mathbb{R}^p$*.*'

   In fact this is a 'characterization' for the notion of subspaces in the following sense:—

   > '*Subspaces of* $\mathbb{R}^p$ *are are those and only those non-empty sets of column vectors with* p *real entries, for which it happens that there is definitely no chance for any linear combination of column vectors belong to such a set to "fall outside" the set concerned.*'

   This is given a precise formulation in the statement of Theorem (3).

8. **Theorem (3). (Characterization for subspaces of** $\mathbb{R}^p$ **in terms of linear combinations.)**

   Suppose $\mathcal{W}$ *be a non-empty set of column vectors with* p *real entries.*

   Then the statements below are logically equivalent:—

   (1) $\mathcal{W}$ *is a subspace of* $\mathbb{R}^p$ *over the reals.*

   (2) *For any positive integer* n*, if* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n \in \mathcal{W}$*, and* $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$*, then* $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n \in \mathcal{W}$*.*

9. **Proof of Theorem (3).**

   Suppose $\mathcal{W}$ be a non-empty set of column vectors with p real entries.

   (a) Lemma (2) informs us that the statement (1) implies the statement (2).

   (b) Suppose the statement (2) holds: *For any positive integer* n*, if* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n \in \mathcal{W}$*, and* $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$*, then* $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n \in \mathcal{W}$*.*
   [We intend to deduce the statement (1): '$\mathcal{W}$ is a subspace of $\mathbb{R}^p$ over the reals.']

   i. [We verify (S2): '*For any* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^p$*, if* $\mathbf{u} \in \mathcal{W}$ *and* $\mathbf{v} \in \mathcal{W}$ *then* $\mathbf{u} + \mathbf{v} \in \mathcal{W}$*.*']
   Pick any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^p$. Suppose $\mathbf{u} \in \mathcal{W}$ and $\mathbf{v} \in \mathcal{W}$.
   By the statement (2), we have $\mathbf{u} + \mathbf{v} = 1 \cdot \mathbf{u} + 1 \cdot \mathbf{v} \in \mathcal{W}$.

   ii. [We verify (S3): '*For any* $\mathbf{u} \in \mathbb{R}^p$*, for any* $\alpha \in \mathbb{R}$*, if* $\mathbf{u} \in \mathcal{W}$ *then* $\alpha\mathbf{u} \in \mathcal{W}$*.*']
   Pick any $\mathbf{u} \in \mathbb{R}^p$. Pick any $\alpha \in \mathbb{R}$. Suppose $\mathbf{u} \in \mathcal{W}$.
   By the statement (2), we have $\alpha\mathbf{u} \in \mathcal{W}$.

   iii. [We verify (S1): '$\mathbf{0}_p \in \mathcal{W}$.']
   By assumption, $\mathcal{W} \neq \emptyset$. There is some column vector with p real entries, say, $\mathbf{t}$, belonging to $\mathcal{W}$.
   Note that $\mathbf{0}_p = 1 \cdot \mathbf{t} + (-1)\mathbf{t}$.
   Recall $\mathbf{t} \in \mathcal{W}$. Then by the statement (2), $1 \cdot \mathbf{t} + (-1)\mathbf{t} \in \mathbf{W}$.
   Hence $\mathbf{0}_p \in \mathbf{W}$.

10. **Theorem (4). ('Extreme' subspaces of** $\mathbb{R}^p$ **over the reals.)**

    (Here p *is assumed to be a positive integer.)*

    (1) $\{\mathbf{0}_p\}$ *constitutes a subspace of* $\mathbb{R}^p$ *over the reals.*

    (2) $\mathbb{R}^p$ *constitutes a subspace of* $\mathbb{R}^p$ *over the reals.*

Proof of Theorem (4). Exercise on the definition for the notion of subspace.

Remark on terminologies and notations.

(a) $\{\mathbf{0}_p\}$ is referred to as the **zero subspace of** $\mathbb{R}^p$, or **trivial subspace of** $\mathbb{R}^p$. (Some people denote it as $\mathcal{O}_p$.)
In contrast to $\{\mathbf{0}_p\}$, every subspace of $\mathbb{R}^p$ other than $\{\mathbf{0}_p\}$ will be referred to as a **non-zero subspace of** $\mathbb{R}^p$, or a **non-trivial subspace of** $\mathbb{R}^p$.

(b) $\mathbb{R}^p$ is referred to as the **improper subspace of** $\mathbb{R}^p$ (though this name is not often used).
In contrast to $\mathbb{R}^p$, every subspace of $\mathbb{R}^p$ other than $\mathbb{R}^p$ itself will be referred to as a **proper subspace of** $\mathbb{R}^p$.

(c) For the above reasons, every subspace of $\mathbb{R}^p$ which is neither $\{\mathbf{0}_p\}$ nor $\mathbb{R}^p$ itself will be referred to as a **non-trivial proper subspace of** $\mathbb{R}^p$.

11. In terms of the notion of subspace, our earlier observations on null space of an arbitrary matrix can be summarized in a one-line result:—

Theorem (5).

Suppose A is an $(m \times p)$-matrix with real entries.

Then $\mathcal{N}(A)$ constitutes a subspace of $\mathbb{R}^p$ over the reals.

Remark.

(a) This is in fact the reason we use the word '*space*' in the phrase '*null space*'.

(b) $\{\mathbf{0}_p\}, \mathbb{R}^p$ can be thought of as null spaces of some appropriate matrices:—

$$\mathcal{N}(I_p) = \{\mathbf{0}_p\}, \qquad \mathcal{N}(\mathcal{O}_{1\times p}) = \mathbb{R}^p$$

12. **Theorem (6).**

Suppose A is an $(m \times p)$-matrix with real entries, and $\mathbf{b}$ is a column vector with m real entries.

Then $\mathcal{S}(A, \mathbf{b})$ is a subspace of $\mathbb{R}^p$ over the reals if and only if $\mathbf{b} = \mathbf{0}_p$.

Proof of Theorem (6). Exercise. (The key is whether $\mathbf{0}_p$ belongs to $\mathcal{S}(A, \mathbf{b})$ or not.)

Remark. In plain words, this result says that the solution set of $\mathcal{LS}(A, \mathbf{b})$ is a subspace of $\mathbb{R}^p$ over the reals exactly when the system $\mathcal{LS}(A, \mathbf{b})$ is homogeneous.

13. **Theorem (7).**

Suppose $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_q$ are column vectors with p real entries. Then $\mathrm{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_q\})$ *constitutes a subspace of $\mathbb{R}^p$ over the reals.*

Proof of Theorem (7). Exercise.

Remark. The validity of Conditions (S1), (S2), (S3) on $\mathrm{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_q\})$ is 'built into' the very definition for the notion of span, through these statements that we have proved very early on in the study of linear combinations:—

(a) $\mathbf{0}_p$ *is a linear combination of* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_q$.

(b) *The sum of any two linear combinations of* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_q$ *is a linear combination of* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_q$.

(c) *Any scalar multiple of a linear combination of* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_q$ *is a linear combination of* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_q$.

In a way, we may think of Theorem (7) as a short-hand for the collection of these three statements.

In fact, more can be said about $\mathrm{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_q\})$:—

> $\mathrm{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_q\})$ *is the subspace of $\mathbb{R}^p$ over the reals that contains as its elements all of* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_q$, *and all linear combinations of* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_q$, *and nothing else.*

This is the essence of Theorem (7).

14. **Theorem (8).**

Suppose $\mathcal{W}$ is a subspace of $\mathbb{R}^p$, and $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_q \in \mathcal{W}$. Then the statements below are logically equivalent:—

(1) $\mathcal{W} = \mathrm{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_q\})$.

(2) *Every column vector belonging to $\mathcal{W}$ is a linear combination of* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_q$.

Remark on terminologies. In the context of this result, when (1) indeed holds under the assumption in the result, people will say:—

- $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_q$ **span** (or **generate**) the subspace $\mathcal{W}$ over the reals.
- $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_q$ are **generators** for the subspace $\mathcal{W}$ over the reals.
- $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_q\}$ is a **spanning set** (or a **generating set**) for the subspace $\mathcal{W}$ over the reals.

Proof of Theorem (8). Exercise. (The argument is no more than repeated honest applications of the definitions for the notion of set equality and for the notion of span, and an application of Lemma (2) at a crucial place.)

15. **Definition. (Intersection of sets of column vectors with the same number of entries.)**

Suppose S, T are sets of column vectors with p real entries. Then we define the **intersection of** S, T *to be the set*

$$\left\{ \mathbf{x} \in \mathbb{R}^p \;\middle|\; \mathbf{x} \in S \quad \text{and} \quad \mathbf{x} \in T \right\}.$$

We denote this set by $S \cap T$.

Remark. What we have done is to apply the 'selection criterion'

$$\text{` } \mathbf{x} \in S \quad \text{and} \quad \mathbf{x} \in T \text{ '}$$

on the set of all column vectors with p real entries. Collected into a set are those and only those column vectors with p real entries which simultaneously belong to S and belong to T. The resultant set is $S \cap T$.

16. **Theorem (9).**

Suppose $\mathcal{V}, \mathcal{W}$ are subspaces of $\mathbb{R}^p$ over the reals. Then $\mathcal{V} \cap \mathcal{W}$ is a subspace of $\mathbb{R}^p$ over the reals.

Proof of Theorem (9).

Suppose $\mathcal{V}, \mathcal{W}$ are subspaces of $\mathbb{R}^p$ over the reals.

(a) [We verify (S1): '$\mathbf{0}_p \in \mathcal{V} \cap \mathcal{W}$.']
Since $\mathcal{V}$ is a subspace of $\mathbb{R}^p$ over the reals, $\mathbf{0}_p \in \mathcal{V}$.
Since $\mathcal{W}$ is a subspace of $\mathbb{R}^p$ over the reals, $\mathbf{0}_p \in \mathcal{W}$.
Therefore $\mathbf{0}_p \in \mathcal{V}$ and $\mathbf{0}_p \in \mathcal{W}$ simultaneously.
Hence, by definition of intersection, $\mathbf{0}_p \in \mathcal{V} \cap \mathcal{W}$.

(b) [We verify (S2): '*For any $\mathbf{t}, \mathbf{u} \in \mathbb{R}^p$, if $\mathbf{t} \in \mathcal{V} \cap \mathcal{W}$ and $\mathbf{u} \in \mathcal{V} \cap \mathcal{W}$ then $\mathbf{t} + \mathbf{u} \in \mathcal{V} \cap \mathcal{W}$.*']
Pick any $\mathbf{t}, \mathbf{u} \in \mathbb{R}^p$. Suppose $\mathbf{t} \in \mathcal{V} \cap \mathcal{W}$ and $\mathbf{u} \in \mathcal{V} \cap \mathcal{W}$.
By definition of intersection, $\mathbf{t} \in \mathcal{V}$ and $\mathbf{t} \in \mathcal{W}$.
By definition of intersection, $\mathbf{u} \in \mathcal{V}$ and $\mathbf{u} \in \mathcal{W}$.
We have $\mathbf{t} \in \mathcal{V}$ and $\mathbf{u} \in \mathcal{V}$. Since $\mathcal{V}$ is a subspace of $\mathbb{R}^p$, we have $\mathbf{t} + \mathbf{u} \in \mathcal{V}$.
We have $\mathbf{t} \in \mathcal{W}$ and $\mathbf{u} \in \mathcal{W}$. Since $\mathcal{W}$ is a subspace of $\mathbb{R}^p$, we have $\mathbf{t} + \mathbf{u} \in \mathcal{W}$.
Now $\mathbf{t} + \mathbf{u} \in \mathcal{V}$ and $\mathbf{t} + \mathbf{u} \in \mathcal{W}$ simultaneously.
Then, by the definition of intersection, $\mathbf{t} + \mathbf{u} \in \mathcal{V} \cap \mathcal{W}$.

(c) [We verify (S3): '*For any $\mathbf{t} \in \mathbb{R}^p$, for any $\alpha \in \mathbb{R}$, if $\mathbf{t} \in \mathcal{V} \cap \mathcal{W}$ then $\alpha\mathbf{t} \in \mathcal{V} \cap \mathcal{W}$.*']
Pick any $\mathbf{t} \in \mathbb{R}^p$. Pick any $\alpha \in \mathbb{R}$. Suppose $\mathbf{t} \in \mathcal{V} \cap \mathcal{W}$.
By definition of intersection, $\mathbf{t} \in \mathcal{V}$ and $\mathbf{t} \in \mathcal{W}$.
We have $\mathbf{t} \in \mathcal{V}$ and $\alpha \in \mathbb{R}$. Since $\mathcal{V}$ is a subspace of $\mathbb{R}^p$, we have $\alpha\mathbf{t} \in \mathcal{V}$.
We have $\mathbf{t} \in \mathcal{W}$ and $\alpha \in \mathbb{R}$. Since $\mathcal{W}$ is a subspace of $\mathbb{R}^p$, we have $\alpha\mathbf{t} \in \mathcal{W}$.
Now $\alpha\mathbf{t} \in \mathcal{V}$ and $\alpha\mathbf{t} \in \mathcal{W}$ simultaneously.
Then, by the definition of intersection, $\alpha\mathbf{t} \in \mathcal{V} \cap \mathcal{W}$.

It follows that $\mathcal{V} \cap \mathcal{W}$ is a subspace of $\mathbb{R}^p$ over the reals.

17. The relevance of the construction of 'intersection' is illustrated by the result below about null spaces.

Theorem (10).

Let A be an $(m \times p)$-matrix with real entries, and B be an $(n \times p)$-matrix with real entries.

Suppose C is the $((m+n) \times p)$-matrix with real entries given by $C = \left[\begin{array}{c} A \\ \hline B \end{array}\right]$.

Then $\mathcal{N}(C) = \mathcal{N}(A) \cap \mathcal{N}(B)$.

Proof of Theorem (10). Exercise on set equality.

Remark. The conclusion in this result is just a very compact way of formulation the apparently 'obvious fact' about the relation amongst solutions of the homogeneous systems $\mathcal{LS}(A, \mathbf{0}_m)$, $\mathcal{LS}(B, \mathbf{0}_n)$, $\mathcal{LS}(C, \mathbf{0}_{m+n})$:—

$\mathbf{t}$ is a solution of the system $\mathcal{LS}(C,\, \mathbf{0}_{m+n})$, which can be explicitly presented the simultaneous linear equations with unknown vector $\mathbf{x}$ as

$$\left\{ \begin{array}{rcl} A\mathbf{x} & = & \mathbf{0}_m \\ B\mathbf{x} & = & \mathbf{0}_n \end{array} \right. ,$$

if and only if $\mathbf{t}$ is simultaneously a solution of $\mathcal{LS}(A,\, \mathbf{0}_m)$ and a solution of $\mathcal{LS}(B,\, \mathbf{0}_n)$.

18. **Definition. (Sum of sets of column vectors with the same number of entries.)**

Suppose S, T are sets of column vectors with p real entries. Then we define the **sum of** *S, T to be the set*

$$\left\{ \mathbf{x} \in \mathbb{R}^p \;\middle|\; \begin{array}{l} \text{There exist some } \mathbf{u} \in S,\ \mathbf{v} \in T \\ \text{such that } \mathbf{x} = \mathbf{u} + \mathbf{v}. \end{array} \right\}.$$

We denote this set by $S + T$.

Remark. What we have done is to apply the 'selection criterion'

$$\text{` } \mathbf{x} \in S \quad \text{and} \quad \mathbf{x} \in T \text{ '}$$

on the set of all column vectors with p real entries. In plain words, this 'selection criterion' reads:—

'$\mathbf{x}$ *can be expressed as the sum of two column vectors, one belonging to S and the other belonging to T.*'

Collected into a set are those and only those column vectors with p real entries, each of which can be expressed as the sum of some column vector belonging to S and some column vector belonging to T. The resultant set is $S + T$.

19. **Theorem (11).**

Suppose $\mathcal{V}, \mathcal{W}$ are subspaces of $\mathbb{R}^p$ over the reals. Then $\mathcal{V} + \mathcal{W}$ is a subspace of $\mathbb{R}^p$ over the reals.

Proof of Theorem (11). Exercise on definition of the notion of subspace.

20. The relevance of the construction of 'sum' is illustrated by the result below about spans.

Theorem (12).

Suppose $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_k, \mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_\ell$ are column vectors with p real entries.

Then $\mathrm{Span}(\{\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_k, \mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_\ell\}) = \mathrm{Span}(\{\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_k\}) + \mathrm{Span}(\{\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_\ell\})$.

Proof of Theorem (12). Exercise on set equality.