4.2 Set equality (for sets of matrices and sets of vectors).

0. *Assumed background.*

   - Whatever has been covered in Topics 1-3.
   - 4.1 *Sets of matrices and sets of vectors.*

   Abstract. We introduce:—

   - the notion of set equality (in the context of sets of matrices, and sets of column/row vectors),
   - how the notion of set equality is used in the formulation of results,
   - how the notion of set equality is used in arguments.

   We also verify a few results about null space, solution set and span whose formulations involve set equalities.

1. **Equality for sets.**

   As with other mathematical objects, we are interested in what we mean by 'equality for such objects'.

   Definition. (Set equality.)

   Suppose S, T are sets. Then we say that S is **equal** *to T, and write $S = T$, if and only if every element of each of S, T belongs to the other of S, T.*

   Remark. The presentation of this 'defining condition' is rather terse. What we really mean is that the equality '$S = T$' holds if and only if both of (†), (‡) are true:—

   (†) *For any object x, if $x \in S$ then $x \in T$.*

   (‡) *For any object y, if $y \in T$ then $y \in S$.*

   The conditions (†), (‡) may be 'combined together' into one condition and (re-)expressed as:—

   (†‡) *For any object z, the statement '$z \in S$' holds if and only if the statement '$z \in T$' holds.*

   Further remark. According to definition (and also according to logic), S is not equal to T if and only if *at least one* of the statements (∼†), (∼‡) hold:—

   (∼†) *There is some object x such that $x \in S$ and $x \notin T$.*

   (∼‡) *There is some object y such that $y \in T$ and $y \notin S$.*

   In this situation, we write $S \neq T$. (Note that there is *no requirement* for both (∼†), (∼‡) to hold.)

2. **Example (1).**

   By inspection on the elements of the respective sets 'as listed', we know these equalities hold:—

   (a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} \right\}.$

   (b) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} \right\}.$

3. As an illustration on the the use of the notion of set equality, we re-formulate some definition concerned with systems of linear equations:—

   Theorem (1). (Re-formulation of consistency of systems of linear equations in set language.)

   Let A be an $(m \times n)$-matrix with real entries.

   (a) *Suppose $\mathbf{b} \in \mathbb{R}^m$. Then:—*

      i. *$\mathcal{LS}(A, \mathbf{b})$ is consistent if and only if $\mathcal{S}(A, \mathbf{b}) \neq \emptyset$.*

      ii. *$\mathcal{LS}(A, \mathbf{b})$ is inconsistent if and only if $\mathcal{S}(A, \mathbf{b}) = \emptyset$.*

   (b) i. *$\mathcal{LS}(A, \mathbf{0}_m)$ has some non-trivial solution with real entries if and only if $\mathcal{N}(A) \neq \{\mathbf{0}_n\}$.*

      ii. *$\mathcal{LS}(A, \mathbf{0}_m)$ has no non-trivial solution with real entries if and only if $\mathcal{N}(A) = \{\mathbf{0}_n\}$.*

4. As another illustration on the use of the notion of set equality, we re-formulate a theoretical result that relates invertibility with systems of linear equations whose coefficient matrices are square matrices.

Theorem (2). (Re-formulation of invertibility in terms of null space and solution set.)

Suppose A is a $(p \times p)$-square matrix with real entries. Then the statements below are logically equivalent:—

(1) *A is invertible.*

(2) $\mathcal{N}(A) = \{\mathbf{0}_p\}$.

Moreover, if either of (1), (2) holds, then, for any $\mathbf{b} \in \mathbb{R}^p$, the equality $\mathcal{S}(A, \mathbf{b}) = \{A^{-1}\mathbf{b}\}$ holds.

Remark. In the original presentation of the result, '$\mathcal{S}(A, \mathbf{b}) = \{A^{-1}\mathbf{b}\}$' is formulated as:—

'*The system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution, namely, $A^{-1}\mathbf{b}$*'.

5. We now illustrate how the notion of set equality is used in the presentation of the full description of solutions for systems of linear equations.

Example (2).

(a) *We solve the system of linear equations $\mathcal{LS}(A, \mathbf{b})$, in which $A, \mathbf{b}$ are given by*

$$A = \begin{bmatrix} 1 & -1 & 2 & -7 \\ 3 & -2 & 6 & -18 \\ -4 & 3 & -7 & 23 \\ 1 & 2 & 0 & 7 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -23 \\ -55 \\ 73 \\ 33 \end{bmatrix}.$$

After some work, we conclude that:—

the one and only one solution with real entries of $\mathcal{LS}(A, \mathbf{b})$ is $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

We may present this conclusion as:—

$$\mathcal{S}(A, \mathbf{b}) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}.$$

(b) *We solve the system of linear equations $\mathcal{LS}(A, \mathbf{b})$, in which $A, \mathbf{b}$ are given by*

$$A = \begin{bmatrix} 1 & 3 & -2 & 3 & 21 \\ 2 & 6 & -3 & 5 & 38 \\ 1 & 3 & -4 & 6 & 33 \\ -2 & -6 & 3 & -6 & -42 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

After some work, we conclude that:—

$\mathcal{LS}(A, \mathbf{b})$ *has no solution.*

We may present this conclusion as:—

$$\mathcal{S}(A, \mathbf{b}) = \emptyset.$$

(c) *We solve the system of linear equations $\mathcal{LS}(A, \mathbf{b})$, in which $A, \mathbf{b}$ are given by*

$$A = \begin{bmatrix} 1 & 3 & 1 & -2 & 1 \\ 1 & 3 & 2 & -3 & -3 \\ 2 & 6 & 1 & -2 & 10 \\ -1 & -3 & -3 & 1 & -5 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -3 \\ -4 \\ -3 \\ -1 \end{bmatrix}.$$

After some work, we conclude that:—

$\mathbf{t}$ *is a solution with real entries of $\mathcal{LS}(A, \mathbf{b})$ if and only if*

there are some real numbers u, v such that $\mathbf{t} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} + u \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + v \begin{bmatrix} -9 \\ 0 \\ 0 \\ -4 \\ 1 \end{bmatrix}$.

We may present this conclusion as:—

$$\mathcal{S}(A, \mathbf{b}) = \left\{ \mathbf{x} \in \mathbb{R}^5 \;\middle|\; \text{There exist some } u, v \in \mathbb{R} \text{ such that } \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} + u \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + v \begin{bmatrix} -9 \\ 0 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}.$$

(d) *We solve the system of linear equations $\mathcal{LS}(A,\,\mathbf{b})$, in which $A, \mathbf{b}$ are given by*

$$A = \begin{bmatrix} 0 & 0 & 2 & 3 & 5 & -7 \\ -1 & 2 & 1 & -1 & 0 & -2 \\ 2 & -4 & -1 & 3 & 2 & 1 \\ 3 & -6 & -1 & 5 & 4 & 0 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 12 \\ 0 \\ 5 \\ 10 \end{bmatrix}.$$

After some work, we conclude that:—

$\mathbf{t}$ is a solution with real entries of $\mathcal{LS}(A,\,\mathbf{b})$ if and only if

$$\text{there are some real numbers } u, v, w \text{ such that } \mathbf{t} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \\ 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + v \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

We may present this conclusion as:—

$$\mathcal{S}(A,\,\mathbf{b}) = \left\{ \mathbf{x} \in \mathbb{R}^6 \;\middle|\; \text{There exist some } u, v, w \in \mathbb{R} \text{ such that } \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \\ 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + v \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(e) *We solve the homogeneous system $\mathcal{LS}(A,\,\mathbf{0}_5)$, in which A is given by*

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & -2 & -6 & 3 & 8 \\ -2 & -4 & 3 & -5 & 6 & 28 & -9 & -18 \\ 1 & 2 & -2 & 4 & -4 & -15 & 7 & 19 \\ -3 & -6 & 5 & -6 & 11 & 73 & -14 & -5 \\ -1 & -2 & 2 & -5 & 4 & 8 & -7 & -27 \end{bmatrix}.$$

After some work, we conclude that:—

$\mathbf{t}$ is a solution with real entries of $\mathcal{LS}(A,\,\mathbf{0}_5)$ if and only if

$$\text{there are some real numbers } u, v, w \text{ such that } \mathbf{t} = u \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + v \begin{bmatrix} -3 \\ 0 \\ -5 \\ -7 \\ -9 \\ 1 \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} 2 \\ 0 \\ 3 \\ -8 \\ -6 \\ 0 \\ -4 \\ 1 \end{bmatrix}.$$

We may present this conclusion as:—

$$\mathcal{N}(A) = \left\{ \mathbf{x} \in \mathbb{R}^8 \;\middle|\; \text{There exist some } u, v, w \in \mathbb{R} \text{ such that } \mathbf{x} = u \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + v \begin{bmatrix} -3 \\ 0 \\ -5 \\ -7 \\ -9 \\ 1 \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} 2 \\ 0 \\ 3 \\ -8 \\ -6 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}.$$

(f) *We want to solve the homogeneous system $\mathcal{LS}(A,\,\mathbf{0}_5)$, in which*

$$A = \begin{bmatrix} 1 & 2 & -5 & 15 \\ -1 & -1 & 3 & -9 \\ 3 & 4 & -10 & 31 \\ 2 & 3 & -8 & 25 \\ 1 & 3 & -4 & 13 \end{bmatrix}.$$

After some work, we conclude that:—

the one and only one solution of $\mathcal{LS}(A,\,\mathbf{0}_5)$ is the trivial solution $\mathbf{0}_4$.

We may present this conclusion as:—

$$\mathcal{N}(A) = \{\mathbf{0}_4\}.$$

6. **Example (3). (Illustration on how the definition for the notion of set equality is used in arguments.)**

Let A be a $(3 \times n)$-matrix with real entries whose first and second rows are labelled A_1, A_2 and whose third row is a row of 0's.

Suppose α, β are real numbers, and B is the $(3 \times n)$-matrix with real entries whose rows from top to bottom are $A_1, A_2, \alpha A_1 + \beta A_2$.

We verify that the equality $\mathcal{N}(A) = \mathcal{N}(B)$:—

(a) [We want to verify the statement (†): '*For any* $\mathbf{x}$, *if* $\mathbf{x} \in \mathcal{N}(A)$ *then* $\mathbf{x} \in \mathcal{N}(B)$.']

Pick any $\mathbf{x}$. Suppose $\mathbf{x} \in \mathcal{N}(A)$.

[We ask: Is it true that $\mathbf{x} \in \mathcal{N}(B)$?
This amounts to verifying: '$B\mathbf{x} = \mathbf{0}_3$.'
Now ask: How does the assumption $A\mathbf{x} = \mathbf{0}_3$ help?]

By the definition of matrix multiplication, we have

$$\begin{bmatrix} A_1\mathbf{x} \\ A_2\mathbf{x} \\ {\mathbf{0}_n}^t\mathbf{x} \end{bmatrix} = \begin{bmatrix} A_1 \\ \hline A_2 \\ \hline {\mathbf{0}_n}^t \end{bmatrix}\mathbf{x} = A\mathbf{x} = \mathbf{0}_3.$$

By the definition of matrix equality, we have $A_1\mathbf{x} = 0$ and $A_2\mathbf{x} = 0$.
Then $(\alpha A_1 + \beta A_2)\mathbf{x} = 0$ also.
Therefore $A_1\mathbf{x} = 0$ and $A_2\mathbf{x} = 0$ and $(\alpha A_1 + \beta A_2)\mathbf{x} = 0$.
By the definition of matrix multiplication and matrix equality, we have

$$B\mathbf{x} = \begin{bmatrix} A_1 \\ \hline A_2 \\ \hline \alpha A_1 + \beta A_2 \end{bmatrix}\mathbf{x} = \begin{bmatrix} A_1\mathbf{x} \\ A_2\mathbf{x} \\ (\alpha A_1 + \beta A_2)\mathbf{x} \end{bmatrix} = \mathbf{0}_3.$$

Hence $\mathbf{x} \in \mathcal{N}(B)$.

(b) [We want to verify the statement (‡): '*For any* $\mathbf{y}$, *if* $\mathbf{y} \in \mathcal{N}(B)$ *then* $\mathbf{y} \in \mathcal{N}(A)$.']

Pick any $\mathbf{y}$. Suppose $\mathbf{y} \in \mathcal{N}(B)$.

[We ask: Is it true that $\mathbf{y} \in \mathcal{N}(A)$?
This amounts to verifying: '$A\mathbf{y} = \mathbf{0}_3$.'
Now ask: How does the assumption $B\mathbf{y} = \mathbf{0}_3$ help?]

By the definition of matrix multiplication, we have

$$\begin{bmatrix} A_1\mathbf{y} \\ A_2\mathbf{y} \\ (\alpha A_1 + \beta A_2)\mathbf{y} \end{bmatrix} = \begin{bmatrix} A_1 \\ \hline A_2 \\ \hline \alpha A_1 + \beta A_2 \end{bmatrix}\mathbf{y} = B\mathbf{y} = \mathbf{0}_3.$$

By the definition of matrix equality, we have $A_1\mathbf{y} = 0$ and $A_2\mathbf{y} = 0$ and $(\alpha A_1 + \beta A_2)\mathbf{y} = 0$.
In particular, $A_1\mathbf{y} = 0$ and $A_2\mathbf{y} = 0$.
Then

$$A\mathbf{y} = \begin{bmatrix} A_1 \\ \hline A_2 \\ \hline {\mathbf{0}_n}^t \end{bmatrix}\mathbf{y} = \begin{bmatrix} A_1\mathbf{y} \\ A_2\mathbf{y} \\ {\mathbf{0}_n}^t\mathbf{y} \end{bmatrix} = \mathbf{0}_3.$$

Hence $\mathbf{y} \in \mathcal{N}(A)$.

Remark. In plain words, and in the language of systems of linear equations, this example informs us:—
In the homogeneous system

$$(T): \quad \left\{ \begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ a_{31}x_1 & + & a_{32}x_2 & + & \cdots & + & a_{3n}x_n & = & 0 \end{array} \right. ,$$

if its third equation

$$a_{31}x_1 \quad + \quad a_{32}x_2 \quad + \quad \cdots \quad + \quad a_{3n}x_n \quad = \quad 0$$

can be 'obtained' as a 'linear combination' of its first and and second equations, in the sense that the row vector

$$\begin{bmatrix} a_{31} & a_{32} & \cdots & a_{3n} \end{bmatrix}$$

is a linear combination of the row vectors

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}, \qquad \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix},$$

then the third equation may be 'ignored'. It will happen that the collection of the solutions of the homogeneous system

$$(S): \quad \left\{ \begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ & & & & & & 0 & = & 0 \end{array} \right.$$

and the collection of the solutions of the homogeneous system (T) are the same as each other.

7. Example (3) can be regarded as a very special instance of a much more general result about null space.

Theorem (3). (Null spaces of row-equivalent matrices.)

Let A, A' be $(m \times n)$-matrices with real entries. Suppose A is row-equivalent to A'. Then $\mathcal{N}(A) = \mathcal{N}(A')$.

Remark. This is how the statement of Theorem (3) links up with the content of Example (2):—

- When $m = 3$ and the third row of A is a linear combination of the first and second rows of A, it happens that A is row-equivalent to the matrix A' whose first and second rows are respectively the same as that of A and whose third row is a row of 0's.

8. Theorem (3) is in fact a special case of a slightly more general result.

Theorem (4). (Solution sets of systems whose coefficients matrices and vectors of constants are row-equivalent under the same sequence of row operations.)

Let A, A' be $(m \times n)$-matrices with real entries, and $\mathbf{b}, \mathbf{b}'$ be column vectors with m real entries.

Suppose $A, \mathbf{b}$ are respectively row-equivalent to $A', \mathbf{b}'$ under the same sequence of row operations.

Then $\mathcal{S}(A, \mathbf{b}) = \mathcal{S}(A', \mathbf{b}')$.

Remark. This is no more than a re-formulation of an earlier result that we have learnt. In that earlier result, whose assumption is the same as that of Theorem (4), we have this conclusion:—

For any column vector $\mathbf{t}$ with n real entries,

$\mathbf{t}$ is a solution of $\mathcal{LS}(A, \mathbf{b})$ if and only if $\mathbf{t}$ is a solution of $\mathcal{LS}(A', \mathbf{b}')$.

But this is simply a 'wordy formulation' of the set equality '$\mathcal{S}(A, \mathbf{b}) = \mathcal{S}(A', \mathbf{b}')$'.

9. As an illustration on how to use the definition of set equality in arguments, and how our knowledge on the relation between invertibility and row-equivalence can be applied, we give a (re-)proof for Theorem (4).

Proof of Theorem (4).

Let A, A' be $(m \times n)$-matrices with real entries, and $\mathbf{b}, \mathbf{b}'$ be column vectors with m real entries.

Suppose $A, \mathbf{b}$ are respectively row-equivalent to $A', \mathbf{b}'$ under the same sequence of row operations.

By assumption, $[\ A \mid \mathbf{b}\ ]$ is row-equivalent to $[\ A' \mid \mathbf{b}'\ ]$.

Then there exist some invertible $(m \times m)$-square matrix H such that $[\ A' \mid \mathbf{b}'\ ] = H[\ A \mid \mathbf{b}\ ]$.

We have $A' = HA$ and $\mathbf{b}' = H\mathbf{b}$.

Moreover, since H is invertible, H^{-1} is well-defined as an $(m \times m)$-square matrix and $H^{-1}H = I_m$.

(a) [We verify the statement (‡): '*For any $\mathbf{t} \in \mathbb{R}^n$, if $\mathbf{t} \in \mathcal{S}(A, \mathbf{b})$ then $\mathbf{t} \in \mathcal{S}(A', \mathbf{b}')$.*']
Pick any $\mathbf{t} \in \mathbb{R}^n$. Suppose $\mathbf{t} \in \mathcal{S}(A, \mathbf{b})$.
Then $A\mathbf{t} = \mathbf{b}$.
Therefore $A'\mathbf{t} = (HA)\mathbf{t} = H(A\mathbf{t}) = H\mathbf{b} = \mathbf{b}'$.
Hence $\mathbf{t} \in \mathcal{S}(A', \mathbf{b}')$.

(b) [We verify the statement (‡): '*For any $\mathbf{s} \in \mathbb{R}^n$, if $\mathbf{s} \in \mathcal{S}(A', \mathbf{b}')$ then $\mathbf{s} \in \mathcal{S}(A', \mathbf{b}')$.*']
Pick any $\mathbf{s} \in \mathbb{R}^n$. Suppose $\mathbf{s} \in \mathcal{S}(A', \mathbf{b}')$.
Then $A'\mathbf{s} = \mathbf{b}'$.
Therefore $A\mathbf{s} = I_m(A\mathbf{s}) = (H^{-1}H)(A\mathbf{s}) = H^{-1}[H(A\mathbf{s})] = H^{-1}[(HA)\mathbf{s}] = H^{-1}(A'\mathbf{s}) = H^{-1}\mathbf{b}' = H^{-1}(H\mathbf{b}) = (H^{-1}H)\mathbf{b} = I_m\mathbf{b} = \mathbf{b}$.
Hence $\mathbf{s} \in \mathcal{S}(A, \mathbf{b})$.

10. **Example (4). (Illustration on how the definition for the notion of set equality is used in arguments.)**

Suppose $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{R}^p$, and $\alpha_1, \alpha_2 \in \mathbb{R}$. Define $\mathbf{v} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2$.

Define $S = \mathrm{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}\})$, $T = \mathrm{Span}(\{\mathbf{u}_1, \mathbf{u}_2\})$. We verify $S = T$:—

(a) [We want to verify the statement (†): '*For any $\mathbf{x} \in \mathbb{R}^p$, if $\mathbf{x} \in S$ then $\mathbf{x} \in T$.*']
Pick any $\mathbf{x} \in \mathbb{R}^p$. Suppose $\mathbf{x} \in S$.

[We ask: Is it true that $\mathbf{x} \in T$?
This amounts to verifying: '*there exist some $\beta_1, \beta_2 \in \mathbb{R}$ such that $\mathbf{x} = \beta_1\mathbf{u}_1 + \beta_2\mathbf{u}_2$.*'
Now ask: Can we name some appropriate real numbers β_1, β_2 satisfying $\mathbf{x} = \beta_1\mathbf{u}_1 + \beta_2\mathbf{u}_2$?
Then ask: How does the assumption '$\mathbf{x} \in S$' help?]

Since $\mathbf{x} \in S$, there exist some $a_1, a_2, c \in \mathbb{R}$ such that $\mathbf{x} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + c\mathbf{v}$.

For the same a_1, a_2, c, we have $\mathbf{x} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + c\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + c(\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2) = (a_1 + c\alpha_1)\mathbf{u}_1 + (a_2 + c\alpha_2)\mathbf{u}_2$.

Since $a_1, a_2, c, \alpha_1, \alpha_2$ are real numbers, $a_1 + c\alpha_1, a_2 + c\alpha_2$ are also real numbers.

Then, by definition, $\mathbf{x} \in T$.

(b) [We want to verify the statement (‡): '*For any* $\mathbf{y} \in \mathbb{R}^p$, *if* $\mathbf{y} \in T$ *then* $\mathbf{y} \in S$.']

Pick any $\mathbf{y} \in \mathbb{R}^p$. Suppose $\mathbf{y} \in T$.

[We ask: Is it true that $\mathbf{y} \in S$?
This amounts to verifying: '*there exist some* $\gamma_1, \gamma_2, \delta \in \mathbb{R}$ *such that* $\mathbf{y} = \gamma_1\mathbf{u}_1 + \gamma_2\mathbf{u}_2 + \delta\mathbf{v}$.'
Now ask: Can we name some appropriate real numbers $\gamma_1, \gamma_2, \delta$ satisfying $\mathbf{y} = \gamma_1\mathbf{u}_1 + \gamma_2\mathbf{u}_2 + \delta\mathbf{v}$?
Then ask: How does the assumption '$\mathbf{y} \in T$' help?]

Since $\mathbf{y} \in T$, there exist some $a_1, a_2 \in \mathbb{R}$ such that $\mathbf{y} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2$.

For the same a_1, a_2, we have $\mathbf{y} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \mathbf{0}_p = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + 0 \cdot \mathbf{v}$.

Note that $a_1, a_2, 0$ are real numbers.

Then, by definition, $\mathbf{y} \in T$.

11. Example (4) is a special case of a more general result about span and linear combinations.

Theorem (5).

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q, \mathbf{v} \in \mathbb{R}^p$.

Suppose $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *over the reals.*

Then $\text{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q, \mathbf{v}\}) = \text{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\})$.

12. **Proof of Theorem (5).**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q, \mathbf{v} \in \mathbb{R}^p$.

Suppose $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ over the reals.

By definition, there exist some $\alpha_1, \alpha_2, \cdots, \alpha_q \in \mathbb{R}$ such that $\mathbf{v} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q$.

Write $S = \text{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q, \mathbf{v}\})$, $T = \text{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\})$.

We verify $S = T$ according to the definition of set equality:—

(a) [We want to verify the statement (†): '*For any* $\mathbf{x} \in \mathbb{R}^p$, *if* $\mathbf{x} \in S$ *then* $\mathbf{x} \in T$.']

Pick any $\mathbf{x} \in \mathbb{R}^p$. Suppose $\mathbf{x} \in S$.

[We ask: Is it true that $\mathbf{x} \in T$?
This amounts to verifying: '*there exist some* $\beta_1, \beta_2, \cdots, \beta_q \in \mathbb{R}$ *such that* $\mathbf{x} = \beta_1\mathbf{u}_1 + \beta_2\mathbf{u}_2 + \cdots + \beta_q\mathbf{u}_q$.'
Now ask: How does the assumption '$\mathbf{x} \in S$' help?]

Since $\mathbf{x} \in S$, there exist some $a_1, a_2, \cdots, a_q, c \in \mathbb{R}$ such that $\mathbf{x} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_q\mathbf{u}_q + c\mathbf{v}$.

For the same $a_1, a_2, \cdots, a_q, c$, we have

$$\begin{aligned}
\mathbf{x} &= a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_q\mathbf{u}_q + c\mathbf{v} \\
&= a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_q\mathbf{u}_q + c(\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q) \\
&= (a_1 + c\alpha_1)\mathbf{u}_1 + (a_2 + c\alpha_2)\mathbf{u}_2 + \cdots + (a_q + c\alpha_q)\mathbf{u}_q
\end{aligned}$$

Since $a_1, a_2, \cdots, a_q, c, \alpha_1, \alpha_2, \cdots, \alpha_q$ are real numbers, $a_1 + c\alpha_1, a_2 + c\alpha_2, \cdots, a_q + c\alpha_q$ are also real numbers.

Then, by definition, $\mathbf{x} \in T$.

(b) [We want to verify the statement (‡): '*For any* $\mathbf{y} \in \mathbb{R}^p$, *if* $\mathbf{y} \in T$ *then* $\mathbf{y} \in S$.']

Pick any $\mathbf{y} \in \mathbb{R}^p$. Suppose $\mathbf{y} \in T$.

[We ask: Is it true that $\mathbf{y} \in S$?
This amounts to verifying: '*there exist some* $\gamma_1, \gamma_2, \cdots, \gamma_q, \delta \in \mathbb{R}$ *such that* $\mathbf{y} = \gamma_1\mathbf{u}_1 + \gamma_2\mathbf{u}_2 + \cdots + \gamma_q\mathbf{u}_q + \delta\mathbf{v}$.'
Now ask: Can we name some appropriate real numbers $\gamma_1, \gamma_2, \cdots, \gamma_q, \delta$ satisfying $\mathbf{y} = \gamma_1\mathbf{u}_1 + \gamma_2\mathbf{u}_2 + \cdots + \gamma_q\mathbf{u}_q + \delta\mathbf{v}$?
Then ask: How does the assumption '$\mathbf{y} \in T$' help?]

Since $\mathbf{y} \in T$, there exist some $a_1, a_2, \cdots, a_q \in \mathbb{R}$ such that $\mathbf{y} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_q\mathbf{u}_q$.

For the same $a_1, a_2, \cdots, a_q$, we have $\mathbf{y} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_q\mathbf{u}_q + \mathbf{0}_p = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_q\mathbf{u}_q + 0 \cdot \mathbf{v}$.

Note that $a_1, a_2, \cdots, a_q, 0$ are real numbers.

Then, by definition, $\mathbf{y} \in T$.

13. The converse of Theorem (5) is also true.

    Theorem (6). (Converse of Theorem (5).)

    Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q, \mathbf{v} \in \mathbb{R}^p$.

    Suppose $\mathrm{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q, \mathbf{v}\}) = \mathrm{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\})$.

    Then $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *over the reals.*

14. We combine Theorem (5) and Theorem (6) into one result:—

    Theorem (7).

    Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q, \mathbf{v} \in \mathbb{R}^p$. *Then the statements below are logically equivalent:—*

    (1) $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *over the reals.*

    (2) $\mathrm{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q, \mathbf{v}\}) = \mathrm{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\})$.

15. **Proof of Theorem (6).**

    Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q, \mathbf{v} \in \mathbb{R}^p$.

    Suppose $\mathrm{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q, \mathbf{v}\}) = \mathrm{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\})$.

    Note that $\mathbf{v} = 0 \cdot \mathbf{u}_1 + 0 \cdot \mathbf{u}_2 + \cdots + 0 \cdot \mathbf{u}_q + 1 \cdot \mathbf{v}$.

    So $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q, \mathbf{v}$.

    Then by definition of span, we have $\mathbf{v} \in \mathrm{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q, \mathbf{v}\})$.

    Therefore, by definition of set equality, we have $\mathbf{v} \in \mathrm{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\})$.

    Hence, by definition of span, $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ over the reals.

16. A key step in the proof of Theorem (7) deserves to be singled out and formulated as a result about the notion of span.

    Lemma (8).

    Suppose $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_m \in \mathbb{R}^p$. *Then each of* $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_m$ *belongs to* $\mathrm{Span}(\{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_m\})$.

17. Applying mathematical induction, we deduce the result below.

    Theorem (9).

    Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q, \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n \in \mathbb{R}^p$. *Then the statements below are logically equivalent:—*

    (1) *Each of* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *over the reals.*

    (2) $\mathrm{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q, \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}) = \mathrm{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\})$.

18. The result below is a consequence of Theorem (9). But it is in fact a 'user-friendly' re-formulation of Theorem (9).

    Theorem (10). (Corollary to Theorem (9).)

    Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q, \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n \in \mathbb{R}^p$. *Then the statements below are logically equivalent:—*

    (1) *Each of* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *over the reals, and each of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *is a linear combination of* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ *over the reals.*

    (2) $\mathrm{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\}) = \mathrm{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\})$.

19. **Proof of Theorem (10).**

    Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q, \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n \in \mathbb{R}^p$.

    (a) Suppose the statement (1) holds:—

    - Each of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ over the reals, and
    - each of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ over the reals.

    Since each of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ over the reals, we have

    $$\mathrm{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\}) = \mathrm{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q, \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}).$$

    Since each of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ over the reals, we have

    $$\mathrm{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}) = \mathrm{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n, \mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\}).$$

Then

$$\begin{aligned}\text{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\}) &= \text{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q, \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}) \\ &= \text{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n, \mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\}) \\ &= \text{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\})\end{aligned}$$

Hence the statement (2) holds.

(b) Suppose the statement (2) holds:—

- $\text{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\}) = \text{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\})$.

By Lemma (8), for each $j = 1, 2, \cdots, q$, the column vector $\mathbf{u}_j$ belongs to $\text{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\})$.
Then by assumption, $\mathbf{u}_j$ belongs to $\text{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\})$.
Now by definition of span, $\mathbf{u}_j$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$.
Repeating the arguments above, we deduce that for each $k = 1, 2, \cdots, n$, the column vector $\mathbf{v}_k$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.

20. **Example (5). (Illustration of the content of Theorem (9) and Theorem (10).)**

(a) $\text{Span}\left(\left\{\begin{bmatrix}1\\3\\5\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix}, \begin{bmatrix}3\\7\\11\end{bmatrix}, \begin{bmatrix}2\\6\\10\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}\right) = \text{Span}\left(\left\{\begin{bmatrix}1\\3\\5\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix}\right\}\right)$.

Reason:—

Each of $\begin{bmatrix}3\\7\\11\end{bmatrix}, \begin{bmatrix}2\\6\\10\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$ is a linear combination of $\begin{bmatrix}1\\3\\5\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix}$. Below is the detail:

$$\begin{bmatrix}3\\7\\11\end{bmatrix} = \begin{bmatrix}1\\3\\5\end{bmatrix} + \begin{bmatrix}2\\4\\6\end{bmatrix}, \qquad \begin{bmatrix}2\\6\\10\end{bmatrix} = 2\begin{bmatrix}1\\3\\5\end{bmatrix}, \qquad \begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}2\\4\\6\end{bmatrix} - \begin{bmatrix}1\\3\\5\end{bmatrix}.$$

(b) $\text{Span}\left(\left\{\begin{bmatrix}1\\3\\5\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix}\right\}\right) = \text{Span}\left(\left\{\begin{bmatrix}3\\7\\11\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}\right)$.

Reason:—

Each of $\begin{bmatrix}3\\7\\11\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$ is a linear combination of $\begin{bmatrix}1\\3\\5\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix}$. Below is the detail:

$$\begin{bmatrix}3\\7\\11\end{bmatrix} = \begin{bmatrix}1\\3\\5\end{bmatrix} + \begin{bmatrix}2\\4\\6\end{bmatrix}, \qquad \begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}2\\4\\6\end{bmatrix} - \begin{bmatrix}1\\3\\5\end{bmatrix}.$$

Also, each of $\begin{bmatrix}1\\3\\5\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix}$ is a linear combination of $\begin{bmatrix}3\\7\\11\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$. Below is the detail:

$$\begin{bmatrix}1\\3\\5\end{bmatrix} = \frac{1}{2}\begin{bmatrix}3\\7\\11\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\1\end{bmatrix}, \qquad \begin{bmatrix}2\\4\\6\end{bmatrix} = \frac{1}{2}\begin{bmatrix}3\\7\\11\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\1\\1\end{bmatrix}.$$