3.4.2 Exercise: Row-equivalence from the point of view of invertible matrices.

In all the questions below, unless otherwise stated:—

- You may take for granted the validity of the result below, where it is relevant and appropriate:—

  Suppose A, B are matrices with p rows. Then the statements below are logically equivalent:—

  (1) *A, B are row-equivalent to each other.*

  (2) *There exists some invertible $(p \times p)$-square matrix G such that $B = GA$.*

  Now suppose any one of the above holds (so that both hold). Then, for the same G, the equality $A = G^{-1}B$ holds.

- You may also take for granted that each matrix is row-equivalent to one and only one reduced row-echelon form.

1. Let x, y, z, u, v, w be real numbers, and $A = \begin{bmatrix} 1 & 1 & x & 1 & 5 \\ 2 & 0 & y & 1 & 6 \\ 3 & -2 & z & 1 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 2 & 0 & u \\ 0 & 1 & 5 & 0 & v \\ 0 & 0 & 0 & 1 & w \end{bmatrix}$.

   Denote the columns of A, from left to right, by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5$.

   Denote the columns of B, from left to right, by $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4, \mathbf{b}_5$.

   It is assumed that A is row-equivalent to B.

   (a) What are the values of x, y, z? Justify your answer.

   (b) Are $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ linearly dependent or linearly independent? Justify your answer.

   (c) What are the values of u, v, w? Justify your answer.

2. (a) Prove the statement ($\sharp$):—

   ($\sharp$) *Let A, B be $(p \times q)$-matrices, and G a $(p \times p)$-square matrix.*
   Suppose $B = GA$, and G is invertible.
   Then for each $j = 1, 2, \cdots, q$, the j-th column of A is all zero if and only if the j-th column of B is all zero.

   (b) Hence, or otherwise, deduce the statement ($\sharp^*$):—

   ($\sharp'$) *Let C, D be $(m \times n)$-matrices, and H a $(n \times n)$-square matrix.*
   Suppose $D = CH$, and H is invertible.
   Then for each $i = 1, 2, \cdots, m$, the i-th row of C is all zero if and only if the i-th row of D is all zero.

   (c) Dis-prove the statement ($\natural$), by providing an appropriate counter-example:—

   ($\natural$) *Let C, D be $(m \times n)$-matrices, and G an $(m \times m)$-square matrix.*
   Suppose $D = GC$, and G is invertible.
   Then for each $i = 1, 2, \cdots, m$, if the i-th row of C is all zero then the i-th row of D is all zero.

   (d) Is the statement ($\natural'$) true? Justify your answer.

   ($\natural'$) *Let A, B be $(p \times q)$-matrices, and H a $(q \times q)$-square matrix.*
   Suppose $B = AH$, and H is invertible.
   Then for each $j = 1, 2, \cdots, q$, if the j-th column of A is all zero then the j-th column of B is all zero.

3. Let A, B, C, D, J, K, L, M be $(n \times n)$-square matrices. Suppose there is a sequence of row operations

   $$[\ A \mid J\ ] \longrightarrow \cdots \longrightarrow [\ B \mid K\ ] \longrightarrow \cdots \longrightarrow [\ C \mid L\ ] \longrightarrow \cdots \longrightarrow [\ D \mid M\ ].$$

   Which of the statements are true? For each of those statements which is true, give a proof. For each of those statements which is true, just write '*The statement concerned is false*'.

   (a) *Suppose A is invertible. Then C is invertible.*

   (b) *Suppose K is invertible. Then M is invertible.*

   (c) *Suppose $C = I_n$. Then A is invertible.*

   (d) *Suppose $A = M = I_n$. Then D, J are invertible, and $DJ = I_n$.*

   (e) *Suppose $J = A^3$ and $K = A^2$ and $L = A$ and $M = I_n$. Then A is invertible and $B = I_n$ and $C = A^{-1}$ and $D = A^{-2}$.*

4. Prove the statement ($\sharp$):—

($\sharp$) *Let A be an $(m \times n)$-matrix. Suppose A' is the reduced row-echelon form which is row-equivalent to A. Then the statements below are logically equivalent:—*

(1) *Every row of A' is non-zero.*

(2) *For any $\mathbf{b}$ with m entries, $\mathcal{LS}(A, \mathbf{b})$ is consistent.*

Remark. It is best to formulate the argument in terms of invertible matrices. At some point of the argument, you may need to observe that there are m pivot columns in A'. They make up each of the m column vectors which are $E_{1,1}^{m,1}, E_{2,1}^{m,1}, E_{3,1}^{m,1}, \cdots, E_{m,1}^{m,1}$.

5. (a) Prove the statement below, with direct reference to the definitions of invertibility and of linear combinations:—

   Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{v}$ be column vectors with 6 entries.

   Suppose G is an invertible (6×6)-square matrix. Then the statements below are logically equivalent:—

   (1) *$\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ with respect to scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_4$.*

   (2) *$G\mathbf{v}$ is a linear combination of $G\mathbf{u}_1, G\mathbf{u}_2, G\mathbf{u}_3, G\mathbf{u}_4$ with respect to scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_4$.*

   (b) Hence, or otherwise, deduce the statement below:—

   Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{v}$ be column vectors with 6 entries.

   Suppose $\mathbf{u}_1', \mathbf{u}_2', \mathbf{u}_3', \mathbf{u}_4', \mathbf{v}'$ are row-equivalent to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{v}$ under the same sequence of row operations. Then the statements below are logically equivalent:—

   (1) *$\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ with respect to scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_4$.*

   (2) *$G\mathbf{v}$ is a linear combination of $G\mathbf{u}_1, G\mathbf{u}_2, G\mathbf{u}_3, G\mathbf{u}_4$ with respect to scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_4$.*

6. We introduce the definition for the notion of canonical matrices:—

   For each integer m, n, for each integer k between 0 and the minimum of m, n, denote by $J_k^{m,n}$ the $(m \times n)$-matrix whose $(1,1)$-th, $(2,2)$-th, ..., (k,k)-th entries are 1 and whose all other entries are 0.

   (We understand $J_0^{m,n}$ as $\mathcal{O}_{m \times n}$.)

   The matrix $J_k^{m,n}$ is called the **canonical** $(m \times n)$**-matrix with** k **non-zero entries**.

   (a) Prove the statement ($\sharp$):—

   ($\sharp$) *Suppose A is a $(p \times q)$-matrix. Then there exist some invertible $(p \times p)$-square matrix G, and some invertible $(q \times q)$-square matrix H, such that the equality $A = GJ_r^{p,q}H$ holds and r is the rank of A.*

   Remark. How do you relate, through matrix equalities, the matrices A, B, C, in which B is the reduced row-echelon form which is row-equivalent to A, and C is the reduced row-echelon form which is row-equivalent to B^t?

   (b) For each the $(p \times q)$-matrix below, denoted by A here, express it as a product of the form $GJ_r^{p,q}H$ in which G, H are some appropriate invertible matrices and r is the rank of A.

   i. $A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 6 & 5 & 6 \end{bmatrix}$.

   ii. $A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 3 & -2 & 1 & 7 \\ -1 & 3 & -5 & 3 \end{bmatrix}$.

   iii. $A = \begin{bmatrix} 0 & 1 & -2 & 1 \\ -1 & -2 & 3 & -4 \\ 2 & 7 & -12 & 11 \end{bmatrix}$.

   iv. $A = \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & -1 & 3 \\ 3 & 1 & 5 & -7 & 1 \end{bmatrix}$.

   v. $A = \begin{bmatrix} 0 & 1 & 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 2 & 3 & 4 \\ -2 & -1 & -3 & 3 & 1 & 3 \end{bmatrix}$.

   vi. $A = \begin{bmatrix} 1 & 2 & 7 & 1 & -1 \\ 1 & 1 & 3 & 1 & 0 \\ 3 & 2 & 5 & -1 & 9 \\ 1 & -1 & -5 & 2 & 0 \end{bmatrix}$.