3.4.1 Uniqueness of reduced row-echelon form row-equivalent to given matrix.

0. The material in this appendix is supplementary.

1. Here we give a proof for the result on the uniqueness of the reduced row-echelon form row-equivalent to any arbitrarily given matrix.

   Theorem (7). (Uniqueness of reduced row-echelon form which is row-equivalent to a given matrix.)

   Suppose that A is a matrix, and B, C are reduced row-echelon forms.

   Further suppose that B is row-equivalent to A, and C is also row-equivalent to A.

   Then $B = C$.

2. **Observations on the statement of Theorem (7), and on how we shall apply mathematical induction to prove Theorem (7).**

   In view of the basic properties of row-equivalence, Theorem (7) is logically equivalent to the statement (†):

   (†) *Suppose B, C are row-echelon forms which are row-equivalent to each other. Then $B = C$.*

   By making a 'slight twist' in the wording, we may re-formulate the statement (†) as the statement (‡):

   (‡) *Let r be a non-negative integer. Suppose B, C are row-echelon forms which are row-equivalent to each other, and suppose the rank of C is r. Then $B = C$.*

   In view of the above, we shall prove Theorem (7) by applying the method of mathematical induction to prove the statement (‡).

3. In the argument for the statement (‡), we take for granted the validity of Theorem (6) and Theorem (9).

   Theorem (6).

   Let B, C be $(p \times q)$-matrix. Denote the j-th columns of B, C by $\mathbf{b}_j, \mathbf{c}_j$ respectively for each $j = 1, 2, \cdots, q$.

   Suppose B is row-equivalent to C. Then the statements below hold:—

   (a) *If $\mathbf{b}_j$ is a linear combination of $\mathbf{b}_{k_1}, \mathbf{b}_{k_2}, \cdots, \mathbf{b}_{k_n}$ with respect to scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$ then $\mathbf{c}_j$ is a linear combination of $\mathbf{c}_{k_1}, \mathbf{c}_{k_2}, \cdots, \mathbf{c}_{k_n}$ with respect to scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$.*

   (b) *If $\mathbf{b}_{k_1}, \mathbf{b}_{k_2}, \cdots, \mathbf{b}_{k_n}$ are linearly dependent with the non-trivial relation $\alpha_1\mathbf{b}_{k_1} + \alpha_2\mathbf{b}_{k_2} + \cdots + \alpha_n\mathbf{b}_{k_n} = \mathbf{0}_p$, then $\mathbf{c}_{k_1}, \mathbf{c}_{k_2}, \cdots, \mathbf{c}_{k_n}$ are linear dependent with the non-trivial relation $\alpha_1\mathbf{c}_{k_1} + \alpha_2\mathbf{c}_{k_2} + \cdots + \alpha_n\mathbf{c}_{k_n} = \mathbf{0}_p$.*

   (c) *If $\mathbf{b}_{k_1}, \mathbf{b}_{k_2}, \cdots, \mathbf{b}_{k_n}$ are linearly independent, then $\mathbf{c}_{k_1}, \mathbf{c}_{k_2}, \cdots, \mathbf{c}_{k_n}$ are linear independent.*

   Theorem (9). ('Linear relations' amongst columns of a reduced row-echelon form.)

   Let C be a reduced row-echelon form with q columns. Denote the j-th column of C by $\mathbf{c}_j$ for each $j = 1, 2, \cdots, q$, and denote the (k, ℓ)-th entry of C by $\gamma_{k\ell}$ for each k, ℓ.

   Denote the rank of C by r, and suppose the pivot columns of C, from left to right, are the d_1-th, d_2-th, ... d_r-th columns of C.
   Then:—

   (a) *$\mathbf{c}_{d_1}, \mathbf{c}_{d_2}, \cdots, \mathbf{c}_{d_r}$ are linearly independent.*

   (b) *For each $j = 1, 2, \cdots, q$, if $\mathbf{c}_j$ is a free column, and the pivot columns strictly to the left of $\mathbf{c}_j$ are the d_1-th, d_2-th, ..., d_h-th columns, then $\mathbf{c}_j$ is a linear combination of $\mathbf{c}_{d_1}, \mathbf{c}_{d_2}, \cdots, \mathbf{c}_{d_h}$, with the linear relation $\mathbf{c}_j = \gamma_{1j}\mathbf{c}_{d_1} + \gamma_{2j}\mathbf{c}_{d_2} + \cdots + \gamma_{hj}\mathbf{c}_{d_h}$.*

   (c) *For each $k = 1, 2, \cdots, r$, the d_k-th column of C, (which is the column vector $\mathbf{c}_{d_k}$,) is not a linear combination of the columns of C strictly to its left.*
   In particular, the d_k-th column of C is not a linear combination of the d_1-th, d_2-th, ..., d_{k-1}-th columns of C.

4. **Proof of the statement (‡).**

   Denote by $P(r)$ the proposition below:—

   '*Suppose B, C are row-echelon forms of the same size which are row-equivalent to each other, and suppose the rank of C is r. Then $B = C$.*'

   We verify $P(0)$:

   Suppose B, C are row-echelon forms of the same size which are row-equivalent to each other, and suppose the rank of C is 0.

[We ask: Is it true that $B = C$?]

Since the rank of C is 0, there is no non-zero row in C. Then C is a zero matrix.

Since B is row-equivalent to C, there is some invertible matrix G such that $B = GC$. Since C is the zero matrix, B is also the zero matrix. Then $B = C$.

It follows that $P(0)$ is true.

Let s be a non-negative integer. Suppose $P(s)$ is true. We deduce that $P(s+1)$ is true (with the help of $P(s)$):

(a) Suppose B, C are row-echelon forms of the same size which are row-equivalent to each other, and suppose the rank of C is s.
Suppose the pivot columns of C are the d_1-th, d_2-th, ..., d_s-th, d_{s+1}-th columns. Denote these column vectors by $\mathbf{c}_{d_1}, \mathbf{c}_{d_2}, \cdots, \mathbf{c}_{d_s}, \mathbf{c}_{d_{s+1}}$.
Note that for each $j = 1, 2, \cdots, s+1$, the j-th entry of $\mathbf{c}_{d_j}$ is 1, and all other entries of $\mathbf{c}_{d_j}$ are 0.

(b) Write $B = [\ B_\sharp \mid B_\natural\ ], C = [\ C_\sharp \mid C_\natural\ ]$, in which:—

- $B_\sharp$ is the matrix consisting of all the columns of B strictly to the left of the d_{s+1}-th column of B,
- $B_\natural$ is the rest of B, starting from the d_{s+1}-th column of B, and ending at the last column of B,
- $C_\sharp$ is the matrix consisting of all the columns of C strictly to the left of the d_{s+1}-th column of C, and
- $C_\natural$ is the rest of C, starting from the d_{s+1}-th column of C, and ending at the last column of C.

By construction, $B_\sharp, C_\sharp$ are reduced row-echelon forms and they are row-equivalent.
Moreover, by construction, the rank of $C_\sharp$ is s.

(c) By $P(s)$, we have $B_\sharp = C_\sharp$. So $B = [\ C_\sharp \mid B_\natural\ ]$.

(d) We are going to show that the d_{s+1}-th column of B is the same as that of C, namely, $\mathbf{c}_{d_{s+1}}$:—

i. By definition, the d_{s+1}-th column of C, which is the last pivot column of C, is not a linear combination of the d_1-th, d_2-th, ..., d_s-th columns of C, which are first s pivot columns of C.

ii. Since B is row-equivalent to C, the d_{s+1}-th column of B, which is the first column of $B_\natural$, is not a linear combination of the d_1-th, d_2-th, ..., d_s-th columns of B, which are respectively the d_1-th, d_2-th, ..., d_s-th columns of $C_\sharp$, and are respectively given by $\mathbf{c}_{d_1}, \mathbf{c}_{d_2}, \cdots, \mathbf{c}_{d_s}$.

iii. Then the d_{s+1}-th column of B has at least one non-zero entry starting from the $(s+1)$-th entry downwards.
[Ask: Can we pinpoint such an entry?]

iv. Recall that the rank of $C_\sharp$ is s. Then all rows in $C_\sharp$ below the s-th row are rows of 0's.
As B is a reduced row-echelon form, it is then along the d_{s+1}-th column of B that the first non-zero entry of a non-zero row of B appears.
Then the d_{s+1}-th column of B is a pivot column of B. It is the $(s+1)$-th pivot column of B (after the d_1-th, d_2-th, ..., d_s-th columns of B), and therefore it is the same as $\mathbf{c}_{d_{s+1}}$.

(e) We are going to complete the argument for the equality $B = C$, by studying the columns of B, C strictly to the right of their respective d_{s+1}-th column:—

i. For each $\ell > d_{s+1}$, denote the respective ℓ-th columns of B, C by $\mathbf{b}_\ell, \mathbf{c}_\ell$.
For each k, denote the respective k-th entries of $\mathbf{b}_\ell, \mathbf{c}_\ell$ by $\beta_{k\ell}, \gamma_{k\ell}$.

ii. Since C is a reduced row-echelon form whose pivot columns are the d_1-th, d_2-th, ..., d_s-th, d_{s+1}-th columns, $\mathbf{c}_\ell$ is the linear combination of the d_1-th, d_2-th, ..., d_s-th, d_{s+1}-th columns of C with respect to the scalars $\gamma_{1\ell}, \gamma_{2\ell}, \cdots, \gamma_{s\ell}, \gamma_{s+1,\ell}$.

iii. Since B is row-equivalent to C, we have $\mathbf{b}_\ell$ is the linear combination of the d_1-th, d_2-th, ..., d_s-th, d_{s+1}-th columns of B (which are now known to be $\mathbf{c}_{d_1}, \mathbf{c}_{d_2}, \cdots, \mathbf{c}_{d_s}, \mathbf{c}_{d_{s+1}}$ respectively) with respect to the scalars $\gamma_{1\ell}, \gamma_{2\ell}, \cdots, \gamma_{s\ell}, \gamma_{s+1,\ell}$.
Then $\mathbf{b}_\ell = \gamma_{1\ell}\mathbf{c}_{d_1} + \gamma_{2\ell}\mathbf{c}_{d_2} + \cdots + \gamma_{s\ell}\mathbf{c}_{d_s} + \gamma_{s+1,\ell}\mathbf{c}_{d_{s+1}} = \mathbf{c}_\ell$.

Hence $B = C$.
It follows that $P(s+1)$ is true.

By the Principle of Mathematical Induction, $P(r)$ is true for any non-negative integer r.