3.3.1 Answers to Exercise.

1. ——

2. (a) $A + aI_n$ is invertible. Reason: $A + aI_n$ has a right inverse.

   The matrix inverse of $A + aI_n$ is given by $\frac{1}{ab}B + \frac{1}{a}I_n$.

   (b) A, B commute with each other.

3. Let A, B be $(n \times n)$-square matrices. Suppose $AB - I_n$ is invertible.

   (a) Verify that $(BA - I_n)[B(AB - I_n)^{-1}A - I_n] = \alpha I_n$, in which α is a number whose value is independent of A, B, n and which you have to give explicitly.

   (b) Is $BA - I_n$ invertible? Justify your answer. If $BA - I_n$ is invertible, also write down the matrix inverse of $BA - I_n$ in terms of A, B.

   (a) $\alpha = 1$.

   (b) $BA - I_n$ is invertible. Reason: $BA - I_n$ has a right inverse.

   The matrix inverse of $BA - I_n$ is given by $(BA - I_n)^{-1} = B(AB - I_n)^{-1}A - I_n$

4. (a) *Comment.*

   Be very patient to expand and collect terms at the appropriate moments.

   Never write 'U^{-1}', 'V^{-1}'. It has not be assumed that $p = q$; when $p \neq q$, it does not make sense to talk about U, V being invertible or not.

   To save time and effort, you may use the logical equivalence between invertibility and existence of one of left/right inverse.

   (b) ——

5. ——

6. ——

7. (a) ——

   (b) $A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -a & 1 & 0 & 0 \\ ab & -b & 1 & 0 \\ -abc & bc & -c & 1 \end{bmatrix}$.

8. Let $A = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 1 & 1 & 3 & 4 & 1 \\ 2 & 1 & 4 & 5 & 0 \\ 3 & 0 & 3 & 3 & 1 \end{bmatrix}$.

   (a) Find a reduced row-echelon form A' which is row-equivalent to A.

   (b) Hence, or otherwise, determine whether A is invertible. Justify your answer with reference to any equivalent formulation for the notion of invertibility.

   (c) Hence, or otherwise, determine whether the rows of A are linearly independent.

   (a) $A' = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

   (b) A is not invertible. Reason: $\mathcal{LS}(A, \mathbf{0}_5)$ has some non-trivial solution (because A' has some free column).

   (c) The rows of A are linearly dependent. Reason: A^t is not invertible (because A is not invertible).

9. ———

10. ———

11. ———

12. ———