3.3.1 Exercise: Various necessary and sufficient conditions for invertibility.

Unless otherwise stated, you may take for granted the validity of any re-formulation for the notion of invertibility. (However, for the purpose of this exercise, do not use the re-formulation in terms of basis, nor the re-formulation in terms of determinants.)

1. By using some appropriate equivalent formulation(s) for the notion of invertibility, or otherwise, show that each of the square matrices below is not invertible:—

(a) $\begin{bmatrix} 1 & 8 & 9 & 1 & 0 & 0 \\ 3 & 6 & 9 & 0 & 1 & 0 \\ 5 & 4 & 9 & 0 & 0 & 1 \\ 2 & 6 & 8 & 1 & 0 & 0 \\ 4 & 4 & 8 & 0 & 1 & 0 \\ 6 & 2 & 8 & 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 8 & 8 & 8 & 4 & 4 & 4 \\ 1 & 2 & 3 & 3 & 2 & 1 \\ 7 & 6 & 5 & 1 & 2 & 3 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$

2. Let A, B be square matrices of size n. Let a, b be numbers. Suppose $AB + bA + aB = \mathcal{O}_{n\times n}$.

   Suppose a, b are both non-zero.

   (a) Is $A + aI_n$ invertible? Justify your answer. If $A + aI_n$ is invertible, also find its matrix inverse and express it in terms of a, b and B.

   (b) Do A, B commute with each other? Justify your answer.

3. Let A, B be $(n \times n)$-square matrices. Suppose $AB - I_n$ is invertible.

   (a) Verify that $(BA - I_n)[B(AB - I_n)^{-1}A - I_n] = \alpha I_n$, in which α is a number whose value is independent of A, B, n and which you have to give explicitly.

   (b) Is $BA - I_n$ invertible? Justify your answer. If $BA - I_n$ is invertible, also write down the matrix inverse of $BA - I_n$ in terms of A, B.

4. Let A be a $(p \times p)$-square matrix, B be a $(q \times q)$-square matrix, U be a $(p \times q)$-matrix, and V be a $(q \times p)$-matrix. Suppose A is invertible. Further suppose $B + BVA^{-1}UB$ is also invertible.

   (a) Show that $A + UBV$ is invertible, and that its matrix inverse is given by

   $$(A + UBV)^{-1} = A^{-1} - A^{-1}UB(B + BVA^{-1}UB)^{-1}BVA^{-1}.$$

   (b) Now also suppose that B is invertible.

   Show that

   $$(A + UBV)^{-1} = A^{-1} - A^{-1}U(B^{-1} + VA^{-1}U)^{-1}VA^{-1}.$$

5. Let $\alpha, \beta, \gamma, \delta$ be numbers.

   (a) Let A be the (3×3)-square matrix given by $A = \begin{bmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{bmatrix}$.

   Prove the statement ($\sharp_3$):—

   ($\sharp_3$) *A is invertible if and only if α, β, γ are pairwise distinct.*

   (b) Let B be the (4×4)-square matrix given by $A = \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 \\ 1 & \beta & \beta^2 & \beta^3 \\ 1 & \gamma & \gamma^2 & \gamma^3 \\ 1 & \delta & \delta^2 & \delta^3 \end{bmatrix}$.

   Prove the statement ($\sharp_4$):—

   ($\sharp_4$) *B is invertible if and only if α, β, γ are pairwise distinct.*

6. Let α, β be numbers, and $C_{\alpha,\beta}$ be the matrix given by $C_{\alpha,\beta} = \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 \\ 0 & 1 & 2\alpha & 3\alpha^2 \\ 1 & \beta & \beta^2 & \beta^3 \\ 0 & 1 & 2\beta & 3\beta^2 \end{bmatrix}$.

   Prove the statement ($\natural$):—

   ($\natural$) $C_{\alpha,\beta}$ is invertible if and only if α, β are distinct.

7. Let a, b, c be numbers, and A be the (4×4)-square matrix given by

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ 0 & b & 1 & 0 \\ 0 & 0 & c & 1 \end{bmatrix}.$$

(a) By considering whether the columns of A are linearly independent or linear dependent, or otherwise, show that A is invertible.

(b) Determine the matrix inverse of A by finding a reduced row-echelon form which is row-equivalent to $[\ A \mid I_4\ ]$.
Remark. This method is the same as solving the system of linear equations encoded in the matrix equation

$$A \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} = I_4$$

with the 16 unknowns which are the x_{ij}'s.)

8. Let $A = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 1 & 1 & 3 & 4 & 1 \\ 2 & 1 & 4 & 5 & 0 \\ 3 & 0 & 3 & 3 & 1 \end{bmatrix}$.

(a) Find a reduced row-echelon form A' which is row-equivalent to A.

(b) Hence, or otherwise, determine whether A is invertible. Justify your answer with reference to any equivalent formulation for the notion of invertibility.

(c) Hence, or otherwise, determine whether the rows of A are linearly independent.

9. Let $\mathbf{t}_1, \mathbf{t}_2$ be column vectors with m entries.
Let A be a (2×2)-square matrix, whose (i, j)-th entry is a_{ij}.
Define $\mathbf{u}_1 = a_{11}\mathbf{t}_1 + a_{21}\mathbf{t}_2$ and $\mathbf{u}_2 = a_{12}\mathbf{t}_1 + a_{22}\mathbf{t}_2$.

(a) Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2\ ]$, and $T = [\ \mathbf{t}_1 \mid \mathbf{t}_2\ ]$.
Verify that $U = TA$.

(b) Suppose $\mathbf{t}_1, \mathbf{t}_2$ are linearly independent.
Prove that the statements below are logically equivalent:—
(1) *The column vectors $\mathbf{u}_1, \mathbf{u}_2$ are linearly independent.*
(2) *The matrix A is invertible.*

10. Let $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4$ be row vectors with m entries.
Let A be a (4×4)-square matrix, whose (i, j)-th entry is a_{ij}.
For each $j = 1, 2, 3, 4$, define $\mathbf{u}_j = a_{1j}\mathbf{t}_1 + a_{2j}\mathbf{t}_2 + a_{3j}\mathbf{t}_3 + a_{4j}\mathbf{t}_4$.

(a) Define $U = \begin{bmatrix} \mathbf{u}_1 \\ \hline \mathbf{u}_2 \\ \hline \mathbf{u}_3 \\ \hline \mathbf{u}_4 \end{bmatrix}$, and $T = \begin{bmatrix} \mathbf{t}_1 \\ \hline \mathbf{t}_2 \\ \hline \mathbf{t}_3 \\ \hline \mathbf{t}_4 \end{bmatrix}$.
Verify that $U = A^t T$.

(b) Suppose $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4$ are linearly independent.
Prove that the statements below are logically equivalent:—
(1) *The row vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are linearly independent.*
(2) *The matrix A is invertible.*

11. Prove the statement below:—
Let B, C be $(m \times n)$-matrices. Define the $(n \times n)$-matrix A by $A = B^t C$.
Suppose $m < n$. Then A is not invertible.

12. By considering the re-formulation of invertibility in terms of linear dependence/independence, or otherwise, prove the statement $(\sharp)$:—

($\sharp$) *Let A be a (4×4)-square matrix, and $\mathbf{u}$ be a column vector with 4 entries.*
Suppose $A^4\mathbf{u} = \mathbf{0}_4$ and $A^3\mathbf{u} \neq \mathbf{0}_4$.
Then the (4×4)-square matrix $[\ \mathbf{u} \mid A\mathbf{u} \mid A^2\mathbf{u} \mid A^3\mathbf{u}\ ]$ is invertible.