3.2.1 Answers to Exercise.

1. (a) Invertible. Matrix inverse: $\begin{bmatrix} -3 & 5 \\ 2 & -3 \end{bmatrix}$.

   (b) Invertible. Matrix inverse: $\begin{bmatrix} -1 & 3/2 \\ 2 & 5/2 \end{bmatrix}$.

   (c) Invertible. Matrix inverse: $\begin{bmatrix} 1/3 & 1/3 \\ -1/9 & 2/9 \end{bmatrix}$.

   (d) Not invertible. No matrix inverse.

   (e) Invertible. Matrix inverse: $\begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$.

   (f) Invertible. Matrix inverse: $\begin{bmatrix} 27 & -16 & 6 \\ 8 & -5 & 2 \\ -5 & 3 & -1 \end{bmatrix}$.

   (g) Invertible. Matrix inverse: $\begin{bmatrix} -16 & -11 & 3 \\ 7/2 & 5/2 & -1/2 \\ -5/2 & -3/2 & 1/2 \end{bmatrix}$.

   (h) Not invertible. No matrix inverse.

   (i) Not invertible. No matrix inverse.

   (j) Invertible. Matrix inverse: $\begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & -1/2 & 0 & 0 \\ -1/5 & 1 & 1/5 & 3/5 \\ 2/5 & -1/2 & -2/5 & -1/5 \end{bmatrix}$.

2. (a)

$$I_4 \xrightarrow{\alpha_{21}R_1+R_2} \xrightarrow{\alpha_{31}R_1+R_3} \xrightarrow{\alpha_{41}R_1+R_4} \xrightarrow{\alpha_{32}R_2+R_3} \xrightarrow{\alpha_{42}R_2+R_4} \xrightarrow{\alpha_{43}R_3+R_4} L(\alpha_{21}, \alpha_{31}, \alpha_{41}, \alpha_{32}, \alpha_{42}, \alpha_{43}).$$

   (b)

$$\begin{aligned} &L(\alpha_{21}, \alpha_{31}, \alpha_{41}, \alpha_{32}, \alpha_{42}, \alpha_{43}) \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \alpha_{43} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \alpha_{42} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \alpha_{32} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \alpha_{41} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \alpha_{31} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ \alpha_{21} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

   (c) $L(\alpha_{21}, \alpha_{31}, \alpha_{32}, \alpha_{41}, \alpha_{42}, \alpha_{43})$ is invertible. Reason: It is a product of six invertible matrices (each of them a row-operation matrix).

3. —

4. (a)

$$A \xrightarrow{\frac{1}{2}R_2} \xrightarrow{\frac{1}{3}R_3} \xrightarrow{\frac{1}{4}R_4} \xrightarrow{\frac{1}{5}R_5} \xrightarrow{\frac{1}{6}R_6} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \xrightarrow{R_1 \leftrightarrow R_3} \xrightarrow{R_4 \leftrightarrow R_5} \xrightarrow{R_5 \leftrightarrow R_6} I_6$$

   (b) $A = M[R_5 \leftrightarrow R_6]M[R_4 \leftrightarrow R_5]M[R_1 \leftrightarrow R_3]M[R_1 \leftrightarrow R_2]M[6R_6]M[5R_5]M[4R_4]M[3R_3]M[2R_2]$

   (c) A is invertible. Reason: A is a product of row-operation matrices, which are invertible.

$$A^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/5 \\ 0 & 0 & 0 & 1/6 & 0 & 0 \end{bmatrix}$$

5. (a) $B = I_4$.

   (b) $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 4 & 1 \end{bmatrix}$.

   $D = I_4$.

   (c) A is invertible, and $A^{-1} = C$.

6. (a) $\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & \frac{3}{2} & 1 & -2 & -\frac{3}{2} \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 1 \end{array}\right]$.

   (b) $A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & -1 & 1 & 1 \\ \frac{3}{2} & 1 & -2 & -\frac{3}{2} \\ -1 & 0 & 1 & 1 \end{bmatrix}$.

   (c) The one and only one solution of $\mathcal{LS}(A,\, \mathbf{b})$ is given by $\begin{bmatrix} \frac{b_1}{2} - \frac{b_4}{2} \\ -b_2 + b_3 + b_4 \\ \frac{3b_1}{2} + b_2 - 2b_3 - \frac{3b_4}{2} \\ -b_1 + b_3 + b_4 \end{bmatrix}$.

   (d) The one and only solution of $\mathcal{LS}\big(A^2,\, e_2\big)$ is given by $\begin{bmatrix} 0 \\ 2 \\ 3 \\ 1 \end{bmatrix}$.

7. (a) A is invertible if and only if $\beta \neq 0$.

   (b) i. $A^{-1} = \begin{bmatrix} 0 & 0 & 1 & \beta \\ 0 & 0 & 0 & 1 \\ 0 & \alpha & -\alpha & \alpha \\ -1 & 1 & -1 & 1 \end{bmatrix}$.

   ii. It is necessary that $\alpha \neq 0$.

8. —